Vol. 73, No. 1, February 2000

MATHEMATICS MAGAZINE

PRINCIPLES

OF THE

DIFFERENTIAL AND INTEGRAL

CALCULUS,

FAMILIARLY ILLUSTRATED, AND APPLIED TO A

VARIETY OF USEFUL PURPOSES.

DESIGNED FOR THE INSTRUCTION OF YOUTH.

BY THE

REV. WILLIAM RITCHIE, LL.D. F.R.S.

PROFESSOR OF NATURAL PHILOSOPHY AT THE ROYAL INSTITUTION OF GREAT BRITAIN, AND PROFESSOR OF NATURAL PHILOSOPHY AND ASTRONOMY IN THE UNIVERSITY OF LONDON.

J'ai toujours été persuadé qu'un livre élémentaire ne peut être jugé que par l'expérience : qu'il faut, pour ainsi dire, l'essayer sur l'esprit des élèves ; et vérifier, par cette épreuve, la bonté des méthodes que l'on a choisies.—BIOT, *Essais de Géométrie.*

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1836.

- The Lengthening Shadow
- Chaotic Results
- Minimum Spanning Trees and Determinants

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The French quotation reads (in paraphrase): *I have always been persuaded that an elementary book can only be judged by experience: it is necessary to try it from the student's point of view; and to verify by this means the merit of the methods that one has chosen.*

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ARTICLES

The Lengthening Shadow: The Story of Related Rates

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Introduction

> A boy is walking away from a lamppost. How fast is his shadow moving?
>
> A ladder is resting against a wall. If the base is moved out from the wall, how fast is the top of the ladder moving down the wall?

Such "related rates problems" are old chestnuts of introductory calculus, used both to show the derivative as a rate of change and to illustrate implicit differentiation. Now that some "reform" texts [**4**, **14**] have broken the tradition of devoting a section to related rates, it is of interest to note that these problems originated in calculus reform movements of the 19th century.

Ritchie, related rates, and calculus reform

Related rates problems as we know them date back at least to 1836, when the Rev. William Ritchie (1790–1837), professor of Natural Philosophy at London University 1832–1837, and the predecessor of J. J. Sylvester in that position, published *Principles of the Differential and Integral Calculus*. His text [**21**, p. 47] included such problems as:

> If a halfpenny be placed on a hot shovel, so as to expand uniformly, at what rate is its *surface* increasing when the diameter is passing the limit of 1 inch and 1/10, the diameter being supposed to increase *uniformly* at the rate of .01 of an inch per second?

This related rates problem was no mere practical application; it was central to Ritchie's reform-minded pedagogical approach to calculus. He sought to simplify the presentation of calculus so that the subject would be more accessible to the ordinary, non-university student whose background might include only "the elements of

Geometry and the principles of Algebra as far as the end of quadratic equations." [**21**, p. v] Ritchie hoped to rectify what he saw as a deplorable state of affairs:

> The Fluxionary or Differential and Integral Calculus has within these few years become almost entirely a science of symbols and mere algebraic formulae, with scarcely any illustration or practical application. Clothed as it is in a transcendental dress, the ordinary student is afraid to approach it; and even many of those whose resources allow them to repair to the Universities do not appear to derive all the advantages which might be expected from the study of this interesting branch of mathematical science.

Ritchie's own background was not that of the typical mathematics professor. He had trained for the ministry, but after leaving the church, he attended scientific lectures in Paris, and "soon acquired great skill in devising and performing experiments in natural philosophy. He became known to Sir John Herschel, and through him [Ritchie] communicated [papers] to the Royal Society" [**24**, p. 1212]. This led to his appointment as the professor of natural philosophy at London University in 1832.

To make calculus accessible, Ritchie planned to follow the "same process of thought by which we arrive at actual discovery, namely, *by proceeding step by step from the simplest particular examples till the principle unfolds itself in all its generality*." [**21**, p. vii; italics in original]

Drawing upon Newton, Ritchie takes the change in a magnitude over time as the fundamental explanatory concept from which he creates concrete, familiar examples illustrating the ideas of calculus. He begins with an intuitive introduction to limits through familiar ideas such as these: (i) the circle is the limit of inscribed regular polygons with increasing numbers of sides; (ii) $1/9$ is the limit of $1/10 + 1/100 + 1/1000 + \cdots$; (iii) $1/2x$ is the limit of $h/(2xh + h^2)$ as h approaches 0. Then—crucial to his pedagogy—he uses an expanding square to introduce both the idea of a function and the fact that a uniform increase in the independent variable may cause the dependent variable to increase at an increasing rate. Using FIGURE 1 to illustrate his approach, he writes:

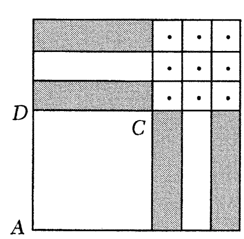

FIGURE 1
An expanding square

> Let AB be the side of a square, and let it increase uniformly by the increments $1, 2, 3$, so as to become $AB + 1$, $AB + 2$, $AB + 3$, etc., and let squares be described on the new sides, as in the annexed figure; then it is obvious that the square on the side $A1$ exceeds that on AB by the two shaded rectangles and the small white square in the corner. The square described on $A2$ has received an increase of two equal rectangles with *three* equal white squares in the corner. The square on $A3$ has received an increase of two equal rectangles and *five* equal small squares. Hence, when the side increases uniformly the area goes on at an increasing rate [**21**, p. 11].

Ritchie continues:

> The object of the differential calculus, is to determine the *ratio* between the rate of variation of the independent variable and that of the function into which it enters [**21**, p. 11].

A problem follows:

> If the side of a square increase uniformly, at what rate does the area increase when the side becomes x? [**21**, p. 11]

His solution is to let x become $x+h$, where h is the rate at which x is increasing. Then the area becomes $x^2 + 2xh + h^2$, where $2xh + h^2$ is the rate at which the area would increase if that rate were uniform. Then he obtains this proportion [**21**, p. 12]:

$$\frac{\text{rate of increase of the side}}{\text{rate of increase of area}} = \frac{h}{2xh + h^2}.$$

Letting h tend to zero yields $1/2x$ for the ratio.

He then turns to this problem:

> If the side of a square increase uniformly at the rate of three feet per second, at what rate is the area increasing when the side becomes 10 feet? [**21**, p. 12]

Using the previous result, he observes that since 1 is to $2x$ as 3 is to $6x$, the answer is 6×10. Then he expresses the result in Newton's notation: If $\dot{x}$ denotes the rate at which a variable x varies at an instant of time and if $u = x^2$, then $\dot{x}$ is to u as 1 is to $2x$, or $u = 2x\dot{x}$.

In his first fifty pages, Ritchie develops rules for differentiation and integration. To illustrate the product rule, he writes:

> If one side of a rectangle vary at the rate of 1 inch per second, and the other at the rate of 2 inches, at what rate is the area increasing when the first side becomes 8 inches and the last 12? [**21**, p. 28]

His problem sets ask for derivatives, differentials, integrals, and the rate of change of one variable given the rate of change of another. Some related rates problems are abstract, but on pages 45–47 Ritchie sets the stage for the future development of related rates with nine problems, most of which concern rates of change of areas and volumes. One was the halfpenny problem; here are three more [**21**, p. 47–48]:

> 25. If the side of an equilateral triangle increase uniformly at the rate of 3 feet per second, at what rate is the area increasing when the side becomes 10 feet?...
>
> 30. A boy with a mathematical turn of mind observing an idle boy blowing small balloons with soapsuds, asked him the following pertinent question:—If the diameter of these balloons increase uniformly at the rate of $1/10$ of an inch per second, at what rate is the internal capacity increasing at the moment the diameter becomes 1 inch?...
>
> 34. A boy standing on the top of a tower, whose height is 60 feet, observed another boy running towards the foot of the tower at the rate of 6 miles an hour on the horizontal plane: at what rate is he approaching the first when he is 100 feet from the foot of the tower?

Since the next section of the book deals with such applications of the calculus as relative extrema, tangents, normals and subnormals, arc length and surface area, Ritchie clearly intended related rates problems to be fundamental, explanatory examples.

Augustus De Morgan (1806–1871) was briefly a professional colleague of Ritchie's at London University. De Morgan held the Chair of Mathematics at London University from 1828 to July of 1831, reassuming the position in October of 1836. Ritchie was appointed in January of 1832 and died in September of 1837. In *A Budget of Paradoxes*, published in 1872, De Morgan wrote [**9**, p. 296]:

> Dr. Ritchie was a very clear-headed man. He published, in 1818, a work on arithmetic, with rational explanations. This was too early for such an improvement, and nearly the whole of his excellent work was sold as waste paper. His elementary introduction to the Differential Calculus was drawn up while he was learning the subject late in life. Books of this sort are often very effective on points of difficulty.

De Morgan, too, was concerned with mathematics education. In *On the Study and Difficulties of Mathematics* [**6**], published in 1831, De Morgan used concrete examples to clarify mathematical rules used by teachers and students. In his short introduction to calculus, *Elementary Illustrations of the Differential and Integral Calculus* [**7**, p. 1–2], published in 1832, he tried to make calculus more accessible by introducing fewer new ideas simultaneously. De Morgan's book, however, does not represent the thoroughgoing reform that Ritchie's does. De Morgan touches on fluxions, but omits related rates problems. In 1836, shortly before Ritchie's death, De Morgan began the serial publication of *The Differential and Integral Calculus*, a major work of over 700 pages whose last chapter was published in 1842. He promised to make "the theory of limits... the sole foundation of the science, without any aid from the theory of series" and stated that he was not aware "that any work exists in which this has been avowedly attempted." [**8**, p. 1] De Morgan was more concerned with the logical foundations of calculus than with pedagogy; no related rates problems appear in the text.

Connell, related rates, and calculus reform

Another reform text appeared shortly after Ritchie's. James Connell, LLD (1804–1846), master of the mathematics department in the High School of Glasgow from 1834 to 1846, published a calculus textbook in 1844 promising "numerous examples and familiar illustrations designed for the use of schools and private students." [**5**, title page] Like Ritchie, Connell complained that the differential calculus was enveloped in needless mystery for all but a select few; he, too, proposed to reform the teaching of calculus by returning to its Newtonian roots [**5**, p. iv]. Connell wrote that he

> ...has fallen back upon the original view taken of this subject by its great founder, and, from the single definition of a rate, has been enabled to carry it out without the slightest assistance from Limiting ratio, Infinitesimals, or any other mode which, however good in itself, would, if introduced here, only tend to mislead and bewilder the student." [**5**, p. v]

To introduce an instantaneous rate, Connell asks the reader to consider two observers computing the speed of an accelerating locomotive as it passes a given point. One notes its position two minutes after it passes the point, the other after one minute; they get different answers for the speed. Instead of considering observations on shorter and shorter time intervals, Connell imagines the engineer cutting off the power at the given point. The locomotive then continues (as customary, neglecting friction) at a constant speed, which both observers could compute. This gives the locomotive's rate, or differential, at that point. Connell goes on to develop the calculus in terms of rates. For example, to prove the product rule for differentials, he considers the rectangular area generated as a particle moves so that its projections along the x- and y-axes move at the rates dx and dy respectively. As with Ritchie, the product rule is taught in terms of an expanding rectangle and rates of change.

Connell illustrates a number of the simpler concepts of the differential calculus using related rates problems. Some of his problems are similar to Ritchie's, but most are novel and original and many remain in our textbooks (punctuation in original):

> 5. A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. [**5**, p. 14]
> 6. One end of a ball of thread, is fastened to the top of a pole, 35 feet high; a person, carrying the ball, starts from the bottom, at the rate of 4 miles per hour, allowing the thread to unwind as he advances; at what rate is it unwinding, when the person is passing a point, 40 feet distant from the bottom of the pole; the height of the ball being 5 feet?...
> 12. A ladder 20 feet long reclines against a wall, the bottom of the ladder being 8 feet distant from the bottom of the wall; when in this position, a man begins to pull the lower extremity along the ground, at the rate of 2 feet per second; at what rate does the other extremity *begin* to descend along the face of the wall?...
> 13. A man whose height is 6 feet, walks from under a lamp post, at the rate of 3 miles per hour, at what rate is the extremity of his shadow travelling, supposing the height of the light to be 10 feet above the ground? [**5**, p. 20–24]

Connell died suddenly on March 26, 1847, leaving a wife and six children. The obituary in the *Glasgow Courier* observed that "he had the rare merit of communicating to his pupils a portion of that enthusiasm which distinguished himself. The science of numbers ... in Dr. Connell's hands ... became an attractive and proper study, and ... his great success as a teacher of children depended on his great attainments as a student of pure and mixed mathematics" [**26**]. It would be interesting to learn of any contact between Ritchie and Connell, but so far we have found none.

The rates reform movement in America

Related rates problems first appeared in America in an 1851 calculus text by Elias Loomis (1811–1889), professor of mathematics at Yale University. Loomis was also concerned to simplify calculus, writing that he hoped to present the material "in a more elementary manner than I have before seen it presented, except in a small volume by the late Professor Ritchie" [**17**, p. iv]. Indeed, the initial portion of

Loomis's text is essentially the same as Ritchie's. Loomis presents ten related rates problems, nine of which are Ritchie's; the one new problem asks for the rate of change of the volume of a cone whose base increases steadily while its height is held constant [**17**, p. 113]. Loomis's text remained in print from 1851 to 1872; a revision remained in print until 1902.

The next text to base the presentation of calculus on related rates was written by J. Minot Rice (1833–1901), professor of mathematics at the Naval Academy, and W. Woolsey Johnson (1841–1923), professor of mathematics at St. John's College in Annapolis. Where Loomis quietly approved the simplifications introduced by Ritchie, Rice and Johnson were much more enthusiastic reformers, drawing more from Connell than from Ritchie:

> Our plan is to return to the method of fluxions, and making use of the precise and easily comprehended definitions of Newton, to deduce the formulas of the Differential Calculus by a method which is not open to the objections which were largely instrumental in causing this view of the subject to be abandoned [**19**, p. 9].

In their 1877 text they derive basic differentiation techniques using rates. Letting dt be a finite quantity of time, dx/dt is the rate of x and "dx and dy are so defined that their ratio is equal to the ratio of the relative rates of x and y" [**20**, p. iv]. This approach has several advantages. First, it allows the authors to delay the definition of dy/dx as the limit of $\Delta y/\Delta x$ until Chapter XI, by which time the definition is more meaningful. Second, "the early introduction of elementary examples of a kinematical character ... which this mode of presenting the subject permits, will be found to serve an important purpose in illustrating the nature and use of the symbols employed" [**20**, p. iv].

These kinematical examples are related rates problems. Rice and Johnson use 26 related rates problems, scattered throughout the opening 57 pages of the text, to illustrate and explain differentiation. Rice and Johnson credit Connell in their preface and some of their problems resemble Connell's. Several other problems are similar to those of Loomis. However, Rice and Johnson also add to the collection of problems:

> A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it? [**20**, p. 28]
>
> Wine is poured into a conical glass 3 inches in height at a uniform rate, filling the glass in 8 seconds. At what rate is the surface rising at the end of 1 second? At what rate when the surface reaches the brim? [**20**, p. 37–38]

After Rice died in 1901, Johnson continued to publish the text until 1909. He was "an important member of the American mathematical scene ... [who] served as one of only five elected members of the Council of the American Mathematical Society for the 1892–1893 term" [**22**, p. 92–93]. The work of Rice and Johnson is likely to have inspired the several late 19th century calculus texts which were based on rates, focusing less on calculus as an analysis of tangent lines and areas and more on "how one quantity changes in response to changes in another." [**22**, p. 92]

James Morford Taylor (1843–1930) at Colgate, Catherinus Putnam Buckingham (1808–1888) at Kenyon, and Edward West Nichols (1858–1927) at the Virginia

Military Institute all wrote texts that remained in print from 1884 to 1902, 1875 to 1889, and 1900 to 1918, respectively. Buckingham, a graduate of West Point and president of Chicago Steel when his text was published was, perhaps, the most zealous of these reformist "rates" authors, believing that limits were problematic and could be avoided by taking rate itself as the primitive concept, much as he believed Newton did [**2**, p. 39].

Newton and precursors of the rates movement

Buckingham was correct that Newton conceived of magnitudes as being generated by motion, thereby linking calculus to kinematics. Newton wrote:

> I consider mathematical quantities in this place not as consisting of very small parts; but as described by a continued motion. Lines are described, and thereby generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines .... These geneses really take place in the nature of things, and are daily seen in the motion of bodies .... Therefore considering that quantities which increase in equal times ... become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions ... and calling these velocities ... *fluxions*. [**3**, p. 413]

Since the 19th century reformers drew on Newton in revising the pedagogy of calculus, one wonders whether rates problems were part of an earlier tradition in England. The first calculus book to be published in English, *A Treatise of Fluxions or an Introduction to Mathematical Philosophy* [**13**] by Charles Hayes (1678–1760), published in 1704, treats fluxions as increments or decrements. Motion is absent and there are no related rates problems. But, in 1706, in the second book published in English, *An Institution of Fluxions* [**10**] by Humphrey Ditton (1675–1715), there are several problems which could be seen as precursors of related rates questions. While Ditton is interested in illustrating ideas of calculus using rates, he sticks to geometrical applications, not mechanical ones. He writes:

> A vast number of other Problems relating to the Motion of Lines and Points which are directly and most naturally solved by Fluxions might have been propos'd to the Reader. But this Field is so large, that 'twill be besides my purpose to do any more upon this Head than only just give some little Hints. [**10**, p. 172]

He gives one worked example. In FIGURE 2, b and c represent the new positions of points B and C respectively:

> If the Line AB, in any moment of Time be supposed to be divided into extream and mean Proportion, as ex. gr in the point C; then the Point A continues fixt, and the Points B and C moving in the direction AB, 'tis requir'd to find the Proportion of the Velocities of the points B and C; so that the flowing line Ab, may still be divided in extream and mean Proportion, e.g. in the Point c." [**10**, p. 171–172]

FIGURE 2
Points moving on a line

In his solution he lets $AB = y$, $AC = x$ and $BC = y - x$ and obtains $\frac{y}{y-x} = \frac{y-x}{x}$ giving $3yx = y^2 + x^2$. He differentiates, obtaining $3y\dot{x} + 3x\dot{y} = 2y\dot{y} + 2x\dot{x}$ which gives $\frac{\dot{x}}{\dot{y}} = \frac{2y-3x}{3y-2x}$. Ditton summarizes by saying, "the velocity of the Increment of the less Segment AC, must be the velocity of the Increment of the whole line AB as $\frac{2AB - 3AC}{3AB - 2AC}$" [**10**, p. 172]. His concern was to express ratios of rates of change of lengths in terms of ratios of lengths within the context of some geometric invariance. He was not seeking to use an equation involving rates of change as a centerpiece of pedagogy, but rather as a straightforward application of the calculus. His work did not lead to the development of such problems. Of the thirteen 18th century English authors surveyed, only William Emerson (1701–1782), writing in 1743, included a related rates problem, but not in a significant way [**11**, p. 108].

In the early 19th century we find scattered related rates problems. There is a sliding ladder problem in a Cambridge collection: "The hypotenuse of a right-angled triangle being constant, find the corresponding variations of the sides." [**25**, p. 678] John Hind's text included one problem: "Corresponding to the extremities of the *latus rectum* of a common parabola, it is required to find the ratio of the rates of increase of the abscissa and ordinate" [**14**, p. 148]. Neither of these problems plays the important pedagogical role that we find in the works of Ritchie or Connell.

The twilight of related rates

Why did so few books illustrate calculus concepts using related rates problems? One reason is that from the beginning of the 18th century to the middle of the 19th century, the foundations of calculus were hotly debated, and Newton's fluxions did not compete very successfully against infinitesimals, limits, and infinite series. Among those who chose to base calculus on fluxions, many still felt uneasy about including kinematical considerations in mathematics. In *A Comparative View of the Principles of the Fluxional and Differential Methods*, Prof. D.M. Peacock wrote that one of the leading objections to the fluxional approach was that "it introduces Mechanical considerations of *Motion*, *Velocity*, and *Time*, foreign to the genius of pure Analytics" [**18**, p. 6]. Such concepts were considered by some to be "inconsistent with the rigour of mathematical reasoning, and wholly foreign to science." [**23**, p. 7]

In England, moreover, resistance to Newton's approach to calculus as well as to the French approach as expressed in Lacroix's textbook [**16**] rested in part on the belief that the purpose of mathematics was to train the mind. That meant doing calculus within a Euclidean framework with a clear focus on the properties of geometrical figures [**1**, p. v–xx].

By the end of the 19th century, most authors were developing calculus on the basis of limits. In the works of Simon Newcomb (1835–1909) and Edward Bowser (1845–1910), for example, related rates problems illustrate the derivative as a rate of change, but the problems are not central. In 1904, William Granville (1863–1943) published his *Elements of the Differential and Integral Calculus*, which remained in print until 1957. This text, which introduced concepts intuitively before establishing analytical arguments, became the standard by which other texts were measured. In the 1941 edition, Granville laid out a method for solving related rates problems, but these problems had now become an end in themselves rather than an exciting and pedagogically important method by which to introduce calculus.

Conclusion

An informal survey of ours suggests that for many teachers these problems have lost their significance. One often hears teachers say that related rates problems are contrived and too difficult for contemporary students. It is thus ironic that such problems entered calculus through reformers who believed, much as modern reformers do, that in order for calculus to be accessible, concrete, apt illustrations of the derivative are necessary. Twilight for our 19th century reformers would have suggested a lengthening, accelerating shadow, not the end of an era. They might well have written:

Related rates, a pump, not a filter; a sail, not an anchor.

NOTE. See `http://www.maa.org/pubs/mm-supplements/index.html` for a more extensive bibliography.

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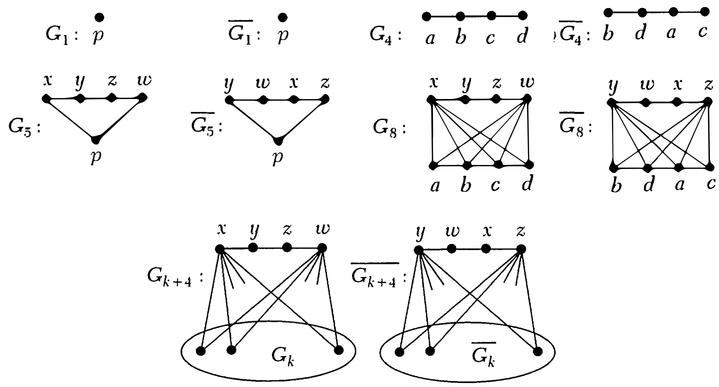
Proof Without Words: Self-Complementary Graphs

A graph is *simple* if it contains no loops or multiple edges. A simple graph $G = (V, E)$ is *self-complementary* if G is isomorphic to its *complement* $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{\{v, w\} : v, w \in V,\ v \neq w, \text{ and } \{v, w\} \notin E\}$. It is a standard exercise to show that if G is a self-complementary simple graph with n vertices, then $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$. A converse also holds, as we shall now show.

THEOREM. *If n is a positive integer and either $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$, then there exists a self-complementary simple graph G_n with n vertices.*

Proof.



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Chaotic Results for a Triangular Map of the Square

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1. Introduction

One-dimensional dynamical systems are well understood, and there are many well-known families of maps which illustrate possible behaviors typical to one dimension. The purpose of this paper is to present an example of a chaotic two-dimensional system (the "triangular map of the square" named in the title). This example illustrates some of the interesting behaviors that are possible in higher dimensions, and so we present it as a springboard that points toward the kind of behaviors one might investigate for other two-dimensional maps, and also toward how one might investigate them.

The intermediate value theorem is the backbone of many of the loveliest and most practical theorems we have for dynamical systems of one variable, whereas the fact that it doesn't hold for functions of more than one variable makes those systems hard to study. A particularly valuable corollary of the intermediate value theorem (due to Brouwer), which we will use several times in Section 3 of this paper, is this:

FIXED POINT THEOREM. *If $f:[a,b]\to\mathbb{R}$ is continuous and either $f([a,b])\subseteq[a,b]$ or $[a,b]\subseteq f([a,b])$, then there is a point $c\in[a,b]$ such that $f(c)=c$.*

We have said that one-dimensional dynamics are well-understood and familiar. In this paper, we will make great use of a typical example of a chaotic dynamical system, the tent map $T: I\to I,\ I=[0,1]$, given by

$$T(x)=\begin{cases}2x & 0\le x\le \frac{1}{2}\\ 2-2x & \frac{1}{2}<x\le 1\end{cases}.$$

As is usual in dynamical systems, we will adopt the notation $T^2=T\circ T$, $T^3=T\circ T\circ T$, and so on. In particular, we will take advantage of the following properties of the tent map:

(1) T is *topologically transitive*: that is, for any pair of non-empty open sets $U,V\subset I$, there is an integer n such that $T^n(U)\cap V\neq\varnothing$;

(2) Even stronger, T is *topologically mixing*: that is, for any pair of non-empty open sets $U,V\subset I$, there are positive numbers N and k such that $T^n(U)\cap T^{-k}(V)\neq\varnothing$ for all $n>N$;

(3) And, in fact, T is *locally eventually onto*: for any non-empty open set $U\subset I$, there is an integer $n>0$ such that $T^n(U)=I$;

(4) The periodic points of T are dense in I; and

(5) If x is a periodic point of T, then $x \in \mathbb{Q}$, and if $x \in \mathbb{Q} \cap I$ then there is some $n \geq 0$ so that $T^n(x)$ is periodic. (It may be that $n = 0$, in which case x is itself a periodic point.)

These standard results can be found in almost every undergraduate course in dynamical systems, and any reader who is familiar with these has sufficient background for the rest of this article. We will use these statements—particularly (3), (4), and (5)—to study our two-dimensional system.

Our map The function $F: I^2 \to I^2$ that we will study throughout the rest of the paper is given by $F:(x, y) \mapsto (T(x), g_x(y))$, where $I^2 = [0, 1] \times [0, 1]$, $T(x)$ is the tent map and

$$g_x(y) = \begin{cases} (\frac{1}{2} + x)y, & \text{for } x \leq \frac{1}{2} \\ (\frac{3}{2} - x)y + (x - \frac{1}{2}), & \text{for } x > \frac{1}{2} \end{cases}.$$

We should make a few remarks about what this function looks like and why we study this function out of all the available possibilities.

The map of F can best be described using pictures. Consider mapping $I^2 = [0, 1] \times [0, 1]$ via F. The effect of $g_x(y)$ is to push down the left hand side of the square toward the x-axis; the right hand side is pushed up toward the line $y = 1$. The tent map stretches the square horizontally to double its original width and then folds the entire left half over the right half. FIGURE 1 shows first the effect just of $g_x(y)$ and then our map F. Hence, F maps the unit square back onto itself.

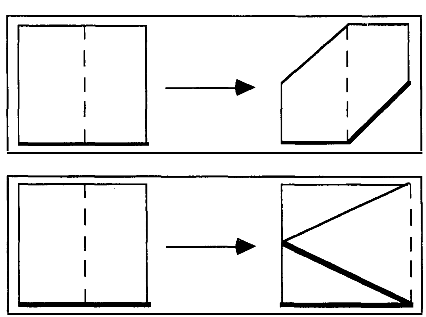

FIGURE 1
Diagram of $(x, y) \mapsto (x, g_x(y))$ and $(x, y) \mapsto (T(x), g_x(y))$.

This is an example of a *triangular map*, so called because in higher dimensions they take the form $F(x, y, z, \dots) = (f(x), g(x, y), h(x, y, z), \dots)$. (These have applications to neural networks.) In two dimensions, our map is known, bizarrely, as a "triangular map of the square." Kolyada **[6]** wrote an extensive paper about these maps, in which he proved (among many other things) two theorems of particular interest to our case:

THEOREM [PERIODIC POINTS]. *If $F(x, y) = (f(x), g(x, y))$ is a continuous function from the unit square into itself and (x, y) is a periodic point of F, then x is a periodic point of f. Conversely, if x is a periodic point of f then there is some $y \in I$ so that (x, y) is a periodic point of F.*

THEOREM [DENSE ORBITS]. *If the orbit of (x, y) under F is dense in I^2, then the orbit of x under f is dense in I.*

The proofs of both of these theorems use the intermediate value theorem and the fixed point theorem; we will prove the result about periodic points in our own specific case in Section 3. The second part of the theorem explains why, if we want F to be chaotic, we must use a chaotic function (such as the tent map) for the first coordinate of our map. If we used a simpler function f, such as a monotone function, then $F=(f, g)$ would not be chaotic, either.

Another nice aspect of this map—one which we will use to get estimates for the sizes of the images of sets—is that $T(x)$ is piecewise linear in x, and that for any fixed x, $g_x(y)$ is linear in y.

Which of the properties that we listed above for T also hold for F? The two questions dynamicists would most like to be able to answer for general maps are ones of transitivity and dense periodic points, since these two criteria are often used to determine whether a function is chaotic (see, e.g., **[5]**). Here, then, are the questions we will ask about our map, along with the answers.

Q: Is F transitive?
A: Yes, in fact it is topologically mixing, a much stronger condition.

Q: Are the periodic points of F dense?
A: Yes, but not as dense as you might think. (We will explain this enigmatic statement in our theorem.)

To be precise, we will prove the following

MAIN THEOREM. *Let $F: I^2 \to I^2$ be defined by $F:(x, y) \mapsto (T(x), g_x(y))$, where*

$$T(x) = \begin{cases} 2x & \text{for } x \le \frac{1}{2} \\ 2-2x & \text{for } x > \frac{1}{2} \end{cases}, \text{ and}$$

$$g_x(y) = \begin{cases} \left(\frac{1}{2}+x\right)y & \text{for } x \le \frac{1}{2} \\ \left(\frac{3}{2}-x\right)y + \left(x-\frac{1}{2}\right) & \text{for } x > \frac{1}{2} \end{cases}.$$

Then F is topologically mixing and the periodic points of (F, I^2) are dense. However, for any given $x \in I$, there is at most one value $y \in I$ so that (x, y) is a periodic point of F.

We will present the proof in Section 3, after a series of lemmas that appear in Section 2.

2. The size of things

What happens to the image of an open set in I^2 when it is mapped by F? From the picture in FIGURE 1, you might expect that the set gets wider and possibly shorter—except that if the set crosses the line $x=\frac{1}{2}$, that part of the set will be mapped to $x=1$ and remain the same height. If we map the image of the set under F yet again, the subsequent image ought to be even wider and shorter than before. This is precisely what the next two lemmas tell us. To simplify notation, we'll use $\|L\|_y$ to denote the height of a vertical line segment $L \subset I^2$, and $\|U\|_x$ to denote the horizontal width of a set $U \subset I^2$. Formally, we define

$$\|U\|_x = \sup\{x \mid (x, y) \in U\} - \inf\{x \mid (x, y) \in U\}.$$

LEMMA 1. *For any connected open set $U \subset I^2$, there is some $n>0$ so that $\|F^n(U)\|_x = 1$.*

LEMMA 2. *The image of the second iterate of any vertical line segment* $L = (\{x\}, [y_0, y_1]) \subset I^2$ *is shorter than L. In particular,* $\|F^2(L)\|_y \le \frac{3}{4}\|L\|_y$.

Lemma 1 is a consequence of the locally eventually onto property of the tent map (the third statement in the introduction). The proof is fairly standard.

Proof of Lemma 1. Pick a positive integer n such that $\frac{1}{2^n} < \|U\|_x$. Then we can find some positive integer k and two values $y_0, y_1 \in I$ such that the points $\left(\frac{k}{2^{n+1}}, y_0\right)$ and $\left(\frac{k+1}{2^{n+1}}, y_1\right)$ are both in U. Without loss of generality, assume $k+1$ is even. Then it is easy to compute that $T^{n+1}\left(\frac{k+1}{2^{n+1}}\right) = 0$ and $T^{n+1}\left(\frac{k}{2^{n+1}}\right) = 1$, so, by the intermediate value theorem, $T^{n+1}\left(\left[\frac{k}{2^{n+1}}, \frac{k+1}{2^{n+1}}\right]\right) = I$. Therefore, $\|F^{n+1}(U)\|_x = 1$. ■

Before we move to the proof of Lemma 2, we will briefly discuss notation. Although we cannot iterate the function g (g takes two variables and returns one), we will nonetheless use $g_x^2(y)$ to denote the second coordinate of $F^2(x, y) = F \circ F(x, y)$.

Proof of Lemma 2. We begin with a line segment $L = \{x\} \times [y_0, y_1]$. Note that $\|L\|_y = y_1 - y_0$. There are four cases to consider:

Case 1: $0 \le x \le \frac{1}{4}$, so that $x \le \frac{1}{2}$ and $T(x) \le \frac{1}{2}$;
Case 2: $\frac{1}{4} \le x \le \frac{1}{2}$, so that $x \le \frac{1}{2}$ and $T(x) \le \frac{1}{2}$;
Case 3: $\frac{1}{2} \le x \le \frac{3}{4}$, so that $x \ge \frac{1}{2}$ and $T(x) \ge \frac{1}{2}$;
Case 4: $\frac{3}{4} \le x \le 1$, so that $x \ge \frac{1}{2}$ and $T(x) \le \frac{1}{2}$.

We prove Case 1; the other cases are similar. Suppose $0 \le x \le 1/4$. Then $g_x(y) = (\frac{1}{2} + x)y$, and hence

$$g_x^2(y) = \left(\tfrac{1}{2} + (2x)\right)\left[\left(\tfrac{1}{2} + x\right)y\right] = \left(\tfrac{1}{4} + \tfrac{3}{2}x + 2x^2\right)y.$$

That is, while $\|L\|_y = (y_1 - y_0)$, we have $\|F^2(L)\|_y = (\frac{1}{4} + \frac{3}{2}x + 2x^2)(y_1 - y_0)$. We could think of $(\frac{1}{4} + \frac{3}{2}x + 2x^2)$ as representing the slope of $g_x^2(y)$ with respect to y. The maximum slope of $g_x(y)$ occurs at the endpoint $x = \frac{1}{4}$; in other words,

$$\|F^2(L)\|_y \le \tfrac{3}{4}(y_1 - y_0) = \tfrac{3}{4}\|L\|. \qquad ■$$

The slope of $g_x^2(y)$ in each region is shown in FIGURE 2 to provide graphical confirmation for all four cases.

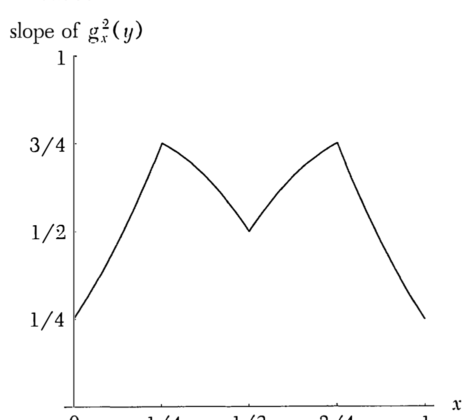

FIGURE 2
Slope of $g_x^2(y)$ for $0 \le x \le 1$.

The above figure leads us to remark that the proof of Lemma 2 could easily be modified to prove:

LEMMA 3. *The second image of any vertical line segment* $L=(\{x\},[y_0,y_1])\subset I^2$ *is not arbitrarily shorter than L. In particular,* $\|F^2(L)\|_y \geq \frac{1}{4}\|L\|_y$.

In order to prove our main theorem, we will have to consider backwards iterates of F as well as forward iterates. What kinds of sets will map into a chosen set? The natural conclusion would be "something that's taller and skinnier." The only problem with this answer is that F is not one-to-one: sometimes the preimage of a set is 2 sets, and we might imagine that two short sets could combine together to produce a tall set. The next lemma says that this is not the case: that even if the preimages of a line segment split into several pieces, at least one of those pieces has to be bigger than the set we started with. The corollary to Lemma 4 says that each set has an eventual preimage which reaches from the top to the bottom of I^2.

LEMMA 4. *Fix a line segment* $L=\{x\}\times[y_0,y_1]$. *There is some* $n>0$ *and some connected set* $K\subset I^2$ *such that* $F^n(K)\subseteq L$ *and* $\|K\|_y \geq \min(\frac{4}{3}\|L\|_y, 1)$.

Proof. Because of the way in which F is defined, mapping a vertical line segment via F^{-1} may result in an image having one or two vertical line segments. If $F^{-1}(L)$ contains a connected component L_1 such that $F(L_1)=L$, then we will say $F^{-1}(L)$ *does not split*. If, on the other hand, $F^{-1}(L)$ consists of two vertical line segments L_1, L_2, such that $F(L_1)\neq L$ and $F(L_2)\neq L$, then we will say $F^{-1}(L)$ *splits*. This latter case happens if y_0 and y_1 lie in different one-to-one regions of I^2—that is, if L crosses both of the lines $y=\frac{1}{2}(x+1)$ and $y=\frac{1}{2}(1-x)$.

Appropriately, we split the proof of the lemma into two segments.

No splitting: If $F^{-1}(L)$ does not split and $F^{-2}(L)$ does not split, then $K=F^{-2}(L)$ is a single vertical line segment, and so it follows directly from Lemma 2 that $\|F^2(K)\|_y \leq \frac{3}{4}\|K\|_y$; that is, $\|K\|_y \geq \frac{4}{3}\|L\|_y$.

At least one splitting: A line segment L splits if it intersects both of the lines $y=\frac{1}{2}(x+1)$ and $y=\frac{1}{2}(1-x)$. The pre-images of these lines are subsets of the lines $y=1$ and $y=0$ respectively. We can write $F^{-1}(L)=L_0\cup L_1$, where L_0 is the component intersecting $y=0$ and L_1 intersects $y=1$ (see FIGURE 3).

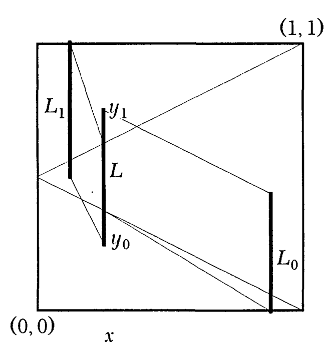

FIGURE 3
The components of $F^{-1}(L)$.

Suppose $F^{-n}(L_0)$ never splits. Then as above, $\|F^{-2m}(L_0)\|_y \geq (\frac{4}{3})^m\|L_0\|_y \geq \frac{4}{3}\|L\|_y$ for sufficiently large m; $K=F^{-2m}(L_0)$ is the line segment we want.

Suppose on the other hand that $F^{-n}(L_0)$ splits for some $n>0$. Then in fact we get a preimage of height one! This is because the lower endpoint of $F^{-n}(L_0)$ lies on the line $y=0$ for all n; a splitting can only happen if the upper endpoint of $F^{-n+1}(L_0)$ lies above the line $y=\frac{1}{2}(x+1)$, whose preimage includes the line $y=1$. This in turn implies (using the intermediate value theorem) that at least one line segment contained in $F^{-n}(L_0)$ looks like $K=\{\tilde{x}\}\times I$, with $T^n(\tilde{x})=x$. So $\|K\|_y=\|F^{-n}(L_0)\|_y=1$.

The remaining case—that $F^{-1}(L_0)$ does not split, but $F^{-2}(L)$ does—is analogous. This completes our proof. ■

COROLLARY. *For any vertical line segment* $L = \{x\} \times [y_0, y_1]$, *there is some* $n > 0$ *and some connected set* $K \subset I^2$ *such that* $F^n(K) \subseteq L$ *and* $\|K\|_y = 1$.

Proof. We will prove the corollary by induction. Pick $m > 0$ such that $\left(\frac{4}{3}\right)^m \|L\|_y \geq 1$. Let $K_0 = L$, and for $i = 1, \dots, m$, let n_i, K_i be as guaranteed by Lemma 4. That is,

$$F^{n_i}(K_i) \subseteq K_{i-1} \quad \text{and} \quad \|K_i\|_y \geq \min\left(\tfrac{4}{3}\|K_{i-1}\|_y, 1\right).$$

Then

$$F^{n_1+n_2+\cdots+n_m}(K_m) \subseteq K_0 = L \quad \text{and} \quad \|K_m\|_y \geq \min\left(\left(\tfrac{4}{3}\right)^m \|L\|_y, 1\right) = 1. \qquad ■$$

3. Putting the pieces together

Now that we have a nice collection of lemmas about images and preimages of sets under F to work with, we are ready to prove our main theorem.

Proof that F is topologically mixing. Pick any two non-empty open sets $U, V \subset I^2$. We want to show that there are positive numbers N and k such that $F^n(U) \cap F^{-1}(V) \neq \varnothing$ for all $n > N$. By the Corollary to Lemma 4, we can select a positive integer k and a set $K \subseteq F^{-k}(V)$ so that $\|K\|_y = 1$. By Lemma 1 we can pick N such that $\|F^N(U)\|_x = 1$. Indeed, $\|F^n(U)\|_x = 1$ for all $n \geq N$ (see FIGURE 4 below). It follows that $F^n(U) \cap F^{-k}(V) \supseteq F^n(U) \cap K \neq \varnothing$.

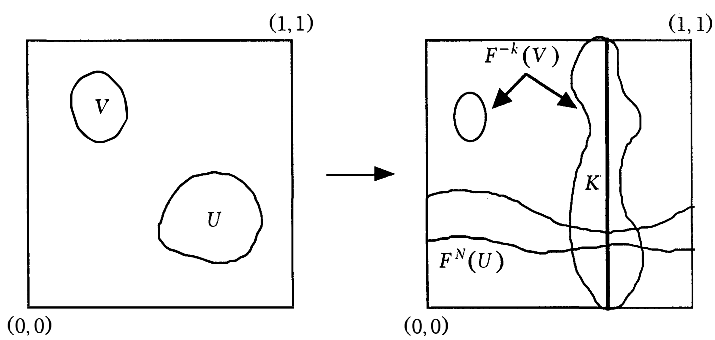

FIGURE 4
Intersection of $F^N(U)$ and $F^{-k}(V)$.

Proof that periodic points of (F, I^2) *are dense.* Fix a non-empty open set $U \subset I^2$. We want to show that there is a point $(x, y) \in U$ that satisfies $F^n(x, y) = (x, y)$ for some $n \neq 0$. Without loss of generality we can assume that U is a rectangle with width $\varepsilon > 0$ and height 3ε.

We will combine the results of Lemma 1, Lemma 2, and topological mixing to select an $n > 0$ which satisfies

(i) $F^n(U)$ intersects non-trivially with the middle $\varepsilon \times \varepsilon$ square in U;

(ii) $\|F^n(U)\|_x = 1$; and

(iii) For every vertical line segment $L \subset F^n(U)$, we have $\|L\|_y < \varepsilon$.

Here's where the fixed point theorem kicks in. Let us use π_x and π_y to denote the projections from I^2 onto the x-axis and the y-axis respectively. By (i) and (ii), we can see that $\pi_x(U) \subset \pi_x(F^n(U)) = I$. Therefore, there is some $x_0 \in \pi_x(U)$ with $\pi_x(F^n(x_0, y)) = x_0$. (This is the same as saying $T^n(x_0) = x_0$, which makes sense—we have just showed that periodic points of the tent map are dense in I.) Fix x_0.

We find y_0 similarly. We will now look only at the part of U which has x_0 as its first coordinate: Let $L_{x_0} = U \cap (\{x_0\} \times I)$. By (i) and (iii) we have $\pi_y(F^n(L_{x_0})) \subset \pi_y(L_{x_0})$. Therefore, the fixed point theorem tells us that there is a fixed point y_0 of $\pi_y(F^n)$. Accordingly, $(x_0, y_0) \in U$ is a fixed point of F^n. This concludes the proof that periodic points are dense. ■

Proof that any vertical line has at most one periodic point. Fix $x_0 \in I$. If x_0 is not a periodic point of (T, I), then clearly there is no $y \in I$ which makes (x_0, y) a periodic point of (F, I^2). Therefore suppose that $T^p(x_0) = x_0$ and that p is the smallest such positive integer. We will show that there exists exactly one $y_0 \in I$ such that $F^p(x_0, y_0) = (x_0, y_0)$; and if $y_1 \neq y_0$, then $F^k(x_0, y_1) \neq (x_0, y_1)$ for $k > 0$.

For fixed x_0, the map $\pi_y(F^p(x_0, *))$ maps I into itself. By the fixed point theorem, it must have at least one fixed point $y_0 \in I$. On the other hand, pick an arbitrary $y \in I$. Then

$$\begin{aligned}\pi_y(F^p(x_0, y)) &= \pi_y \circ F^{p-1}(F(x_0, y))\\ &= \cdots\\ &= g_{T^{p-1}(x_0)}\Big(g_{T^{p-2}(x_0)}\Big(\cdots\big(g_{x_0}(y)\big)\cdots\Big)\Big).\end{aligned}$$

Since $\frac{\partial}{\partial y} g_x(y) \le 1$ for any $x \in I$, we know that $\pi_y(F^p(x_0, *))$ is a contraction: it has no more than one fixed point, which implies (x_0, y_0) is the only periodic point of F on the line segment $\{x_0\} \times I$. ■

The last thing we would like to do in this section is to demonstrate how to find periodic points of (F, I^2). It is clear upon inspection that there is a fixed point at the origin: $F(0,0) = (0,0)$. It is less clear, but still true, that there is a second fixed point at $(\frac{2}{3}, 1)$. The astute reader will notice that $x = 0$ and $x = \frac{2}{3}$ are the only fixed points of the tent map.

In general, if we want to find periodic points of (F, I^2) we begin with the tent map and then "lift" that orbit up into the square. For example, suppose we want to find a period-3 point of (F, I^2). First choose a period-3 orbit of (T, I), say $\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\}$. (We found this orbit by solving $x = T^3(x) = 2 - 2(2(2x))$.) Then we plug $(\frac{2}{9}, y)$ into our triangular map. The first iterate is:

$$F(\tfrac{2}{9}, y) = (\tfrac{4}{9}, (\tfrac{1}{2} + \tfrac{2}{9})y) = (\tfrac{4}{9}, (\tfrac{13}{18})y);$$

the second iterate is:

$$F^2(\tfrac{2}{9}, y) = F(\tfrac{4}{9}, (\tfrac{13}{18})y) = (\tfrac{8}{9}, (\tfrac{1}{2} + \tfrac{4}{9})(\tfrac{13}{18})y) = (\tfrac{8}{9}, (\tfrac{221}{324})y);$$

and the third is:

$$F^3(\tfrac{2}{9}, y) = F(\tfrac{8}{9}, (\tfrac{221}{324})y) = (\tfrac{2}{9}, (\tfrac{3}{2} - \tfrac{8}{9})(\tfrac{221}{324})y + (\tfrac{8}{9} - \tfrac{1}{2})) = (\tfrac{2}{9}, (\tfrac{2431}{5832})y + \tfrac{7}{18}).$$

Setting $y = (\frac{2431}{5832})y + \frac{7}{18}$, we can simplify to get $y = \frac{2268}{3401}$. Thus, one of the period-3 orbits of (F, I^2) is $\{(\frac{2}{9}, \frac{2268}{3401}), (\frac{4}{9}, \frac{1638}{3401}), (\frac{8}{9}, \frac{1547}{3401})\}$. The reader can check that the other is $\{(\frac{2}{7}, \frac{171}{1457}), (\frac{4}{7}, \frac{1881}{20{,}398}), (\frac{8}{9}, \frac{44{,}851}{285{,}572})\}$. The numbers involved are large, but the computations are otherwise straightforward.

4. Conclusion, or what next?

We believe that triangular maps provide a rich source of easily accessible open questions for undergraduates—indeed, this paper is the outcome of an undergraduate research project undertaken during the junior year of one of the authors of this paper. We approached the subject via the question:

> *If a continuous triangular map—that is, a map of the form* $F(x, y) = (f(x), g(x, y))$*—has dense periodic points, and if every iterate of that map is transitive, then is the map topologically mixing?*

Indeed, if we replace "triangular map" by a "map $f: X \to X$", then this question has been proved in the affirmative for $X = I$ [**1**], for $X = S^1$ [**3**], and for X a one-dimensional branched manifold [**2**]. This question is widely known to be true for subshifts of finite type, although there is a counterexample for more general shift maps [**4**]. However, the question is still open for higher-dimensional maps.

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Letter to the editor (1949)

Dear editor,

The enclosed check is to renew my subscription. I want to thank you for two of the things you have done with the *Magazine*.

First, I am glad you try to make the articles intelligible, and not a mere display of the learning of their authors.

Second, I am glad you are helping accustom the mathematical fraternity to the appearance of pages done by offset from typed material.

Very truly yours, etc.

On Minimum Spanning Trees and Determinants

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1. Introduction

In the 1840's, Kirchhoff, while putting forth his defining work on circuit theory, was simultaneously pioneering the theory of graphs, a development which occupies a central place in circuit theory. Using algebraic methods, Kirchhoff **[3]** found a way to determine whether a set of edges in a connected graph was a *spanning tree*, that is, a tree containing all the vertices of the graph. In his setting, the spanning trees correspond to certain nonsingular submatrices of a matrix constructed from the incidence matrix of the graph.

Over time, graphs came to be treated as discrete structures as well, consisting of finite sets of vertices and edges, without the algebraic structure that Kirchhoff used. Later in the century, attention was given to weighted graphs, graphs whose edges have been assigned real values. A classical problem in weighted graphs is this: Given a connected weighted graph, find all the spanning trees that have the minimum edge weight sum. These trees are called *minimum spanning trees*, and an algorithm for finding one was given as early as 1926 by Boruvka, whose work is discussed in **[2]**. In 1956 Kruskal **[4]** developed another algorithm for finding a minimum spanning tree; it is perhaps the most famous of all such algorithms. Minimum spanning trees have many applications; some of the most obvious ones deal with the minimizing of cost, such as the cost of building pipelines to connect storage facilities.

We will begin by outlining the results of Kirchhoff and Kruskal. Although both approaches have to do with spanning trees in graphs, they appear to have little else in common. We show that there is indeed a more profound connection, which we develop here. In particular, we will develop an algorithm that uses Kirchhoff's determinants to generate the set of all minimum spanning trees in a weighted graph. In the process, the linear algebra and the discrete mathematics come together in a pleasing, intriguing, and accessible way, and it is hoped that students may be encouraged to follow this expository article with further exploration.

2. Kirchhoff's theorem

If a graph G has n vertices and b edges, the *incidence matrix* of G is the $n \times b$ matrix in which each column (edge) has exactly two non-zero entries, $+1$ and -1, indicating the two vertices for that edge. (The signs, which can be placed arbitrarily, effectively put a direction on each edge, but this direction is not relevant to what follows.) We assume that G has no loops.

For a connected graph G, the rank of the incidence matrix is $n-1$, and the deletion of any row leaves a matrix of rank $n-1$. We delete the last row to obtain an $(n-1) \times b$ matrix A, called the *reduced incidence matrix*. The determinant of any square submatrix of A is either zero or ± 1. An $(n-1) \times (n-1)$ submatrix of A is non-singular if and only if its columns correspond to the edges of a spanning tree of G. A thorough development of these properties can be found in [**1**], but all are straightforward and make good exercises.

Kirchhoff's observation becomes our first theorem.

THEOREM 1. *The spanning trees of the connected graph G are the nonsingular $(n-1)\times(n-1)$ submatrices of the reduced incidence matrix A, and the determinants of these submatrices are all* ± 1.

Kirchhoff's celebrated formula states that $\det(AA^T)$ is equal to the number of spanning trees in G. This formula, which follows from Theorem 1 and the Binet-Cauchy theorem on determinants, has gathered much attention but does not shed as much light as Theorem 1 itself.

3. Kruskal's algorithm

There are many algorithms for finding a minimum spanning tree in a weighted graph G. Kruskal's algorithm is perhaps the most widely known.

KRUSKAL'S ALGORITHM. *Choose any unchosen edge of lowest weight that does not create a cycle with the chosen edges, and continue until no more edges are available.*

The graph in Figure 1a has four different minimum spanning trees, and Kruskal's algorithm can be executed in such a way as to yield any one of them.

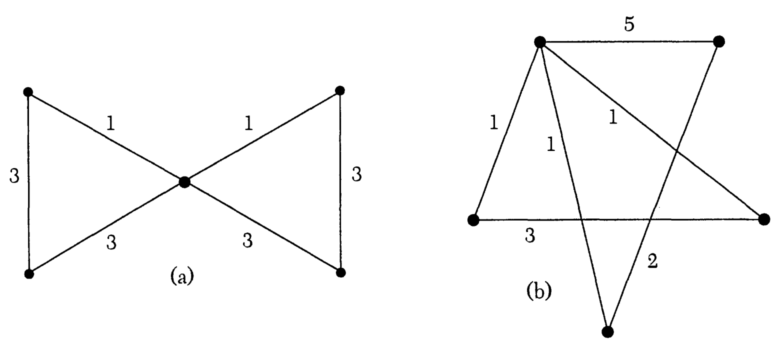

FIGURE 1

In the next section we use Kruskal's algorithm to connect Kirchhoff's determinants and the set of minimum spanning trees in G. First, we need to generalize Kirchhoff's reduced incidence matrix to weighted graphs. Let w be the weight function on the edges of G. For the remainder of this article we assume that w has positive integer values.

We begin by altering the incidence matrix in such a way that the two entries in the column corresponding to the edge e are $\pm w(e)$ instead of ± 1. For the same reasons as before, a set H of $n-1$ edges of G will be a spanning tree if and only if $\det M(H) \neq 0$, where $M(H)$ is the $(n-1)\times(n-1)$ submatrix of the reduced incidence matrix A whose columns correspond to the edges of H. It is also clear that, for any spanning tree H, $\det M(H) = \pm\prod$, where $\prod$ is the product of the weights of the edges in the tree. Fortunately, these determinants can be used to identify the minimum spanning trees of G, as we will see later. For the time being, we will put the matrix A aside and consider geometric properties instead.

In the weighted graph G, suppose T is a minimum spanning tree and f is an edge of G not in T. Suppose further that e lies on the unique simple path (no repeated edges or vertices) in T joining the vertices of f, and that $w(e) = w(f)$. Then one can replace e by f to get another minimum spanning tree. We call this move an *equal edge replacement* in T; others call it an exchange of weight zero.

THEOREM 2. *If T and S are minimum spanning trees in the weighted graph G, then there is a finite sequence of equal edge replacements that start with T and end with S.*

Theorem 2 should perhaps be the first result that students encounter. Yet we have seen it stated only once in a text **[5]**, and there only implicitly, as an exercise.

Every weighted graph has an *edge weight spectrum*, the set of edge weights listed with multiplicities. The following result is an immediate corollary to Theorem 2:

THEOREM 3. *Any two minimum spanning trees in G have the same edge weight spectrum.*

The constancy of the edge weight spectrum can be used to great advantage in studying minimum spanning trees, so it is puzzling that so basic a fact has not made its way into textbooks on discrete mathematics. When the author first began to teach this subject to undergraduates, he naively imagined that one might find two trees that achieved the minimum edge sum using different summands. But it cannot be done.

Proof of Theorem 2. Let k be the smallest integer such that T and S have different sets of edges of weight k. Thus, if an edge of G has weight less than k, then it belongs to T if and only if it belongs to S. Without loss of generality, assume that e is an edge of weight k belonging to T but not to S. In $S \cup \{e\}$ a cycle is formed by e and the unique simple path P joining the vertices of e in S. Every edge on this cycle has weight less than or equal to k, for otherwise, we could replace some edge of greater weight on P by e to obtain a spanning tree with smaller edge sum than S, contrary to hypothesis. Some edge on P must not belong to T, for otherwise, $P \cup \{e\}$ would be a cycle in T. Such an edge must have weight k, for otherwise its weight is less than k and it would belong to T by the choice of k. Choose such an edge f; that is, f is on P, f does not belong to T and $w(f) = k$. Replace f by e in S to get another tree $S' = (S - f) \cup \{e\}$. Now S' is a minimum spanning tree with one more edge in common with T than S. Continue this process (or use induction) to complete the proof. □

Certain elementary results about minimum spanning trees follow effortlessly from Theorem 3. Here are two:

(a) *If all the edges of the connected graph G have different weights, then G has exactly one minimum spanning tree.*
(b) *Every minimum spanning tree in G can be obtained as the output of Kruskal's algorithm.*

4. Determinants

THEOREM 4. *A spanning tree in the weighted graph G has the minimum edge* sum *if and only if it has the minimum edge* product.

To help the reader generate the appropriate surprise at this result, we observe that it is easy to find a weighted graph with two spanning trees T and S, such that T has the smaller edge sum and S has the smaller edge product. The weighted graph in

Figure 1b has two such trees, and the reader may enjoy finding them. But if a tree has the *minimum* edge sum, then it must have the minimum edge product, and conversely.

A short proof of Theorem 4 can be had by replacing each edge weight by its logarithm. Doing so leaves the order of weights unchanged. Since Kruskal's algorithm respects only the ordering of the edges by weight, a spanning tree in G with the old weights can be selected by Kruskal's algorithm if and only if that same tree can be selected with the new weights. But Kruskal's algorithm generates *all* the minimum spanning trees in G, so the set of minimum spanning trees with the new weights is the same as with the old. However, minimizing the edge sum of a tree using the new weights is equivalent to minimizing its edge product using the old weights. Thus, under the old weights, a spanning tree has the minimum edge sum if and only if it has the minimum edge product.

Another proof, less elegant but perhaps more revealing, uses Kruskal's algorithm and the constancy of the edge weight spectrum. Suppose T has the minimum edge product but not the minimum edge sum. Let S be a tree with the minimum edge sum. Then the edge weight spectra of T and S disagree. Let k be the smallest weight at which they disagree.

T cannot have more edges of weight k than S. If T did have more, we could execute Kruskal's algorithm as follows:

Choose the edges of T up through weight $k-1$. (The spectrum of these edges agree with the spectrum of S, and so they can be chosen consistently with Kruskal's algorithm.)

Next, choose the edges of T of weight k. (They don't create a cycle with the previously chosen edges of T.)

Finally, finish Kruskal's algorithm in any acceptable way. The result is a minimum spanning tree, but it differs from S in the number of edges of weight k. This contradicts Theorem 3.

So T must have fewer edges of weight k than S. Let $T(k)$ be the set of edges of T with weight less than or equal to k. Then $T(k)$ can be chosen at the front end of Kruskal's algorithm, and since the spectrum has not yet been used up at weight k, there remains at least one edge e of G of weight k that does not make a cycle with $T(k)$. The edge e does, however, make a cycle with T, and so there is some edge on that cycle with greater weight than e. Replace it with e to get a tree with smaller edge product, contrary to hypothesis. Thus, if T has the minimum edge product, it has the minimum edge sum.

Conversely, let T have the minimum edge sum. Let S be a tree with the minimum edge product. By the preceding paragraph, S has the minimum edge sum as well, and therefore has the same edge weight spectrum as T. Thus T has the minimum edge product. □

Returning now to Kirchhoff's determinants, we can quickly see that these algebraic objects lead us to the minimum spanning trees of G.

THEOREM 5. *Let H be a set of $n-1$ edges of the weighted graph G, and let $M(H)$ be the associated $(n-1)\times(n-1)$ submatrix of A. Then H is a minimum spanning tree if and only if $|\det M(H)|$ is minimal among all nonsingular $(n-1)\times(n-1)$ submatrices of A.*

To establish this theorem, recall that if H is not a tree, then $\det M(H)=0$, and if H is a tree, then $|\det M(H)|$ is the product of the edge weights of H. Thus, H has the minimum nonzero $|\det M(H)|$ if and only if H is a spanning tree with the minimum edge product, and therefore, by Theorem 4, with the minimum edge sum also. □

5. An algorithm for finding all minimum spanning trees

To find all the minimum spanning trees of G, we could search the matrix A for those submatrices with the minimum determinant. This would require evaluating $\binom{b}{n-1}$ determinants. Fortunately, there are more efficient ways to do this search, which we now develop. (The remaining results appear in **[6]**.)

We begin by arbitrarily selecting a reference tree $\mathscr{T}$. Certain edges of G belong to some minimum spanning tree while others do not. We can determine which are which by referring to $\mathscr{T}$.

THEOREM 6. *An edge e of G belongs to some minimum spanning tree if and only if the following condition holds*:

Either e belongs to $\mathscr{T}$, or there is an edge f on the unique path in $\mathscr{T}$ joining the vertices of e such that $w(f) = w(e)$.

Proof. The "if" part is trivial. The "only if" part is not; a proof appears in **[6]**.

Edges that fail to meet this condition can be deleted from G, since their deletion does not disconnect G and they will not be chosen in any search for a minimum spanning tree. Therefore we assume without loss of generality that all edges of G belong to some minimum spanning tree. Next, we partition the edges of G into equivalence classes.

Before defining the equivalence relation, we observe that every equal edge replacement determines a 1–1 function from the edges of T onto the edges of S, where T is a minimum spanning tree and S is the tree that results by replacing some edge e of T by some suitable edge f of equal weight. This function is the identity on all the edges of T except e, and it maps e to f. If $S_0, S_1, \ldots, S_n$ is a sequence of minimum spanning trees resulting from a finite sequence of equal edge replacements, these maps can be composed to give a 1–1 correspondence from the edges of S_0 to the edges of S_n.

For edges e and f, define $e \sim f$ if either $e = f$ or there is a sequence $S_0, S_1, \ldots, S_n$ of equal edge replacements such that e is an edge of S_0, f is an edge of S_n, and the image of e under the composition map is f.

It is easy to see that $\sim$ is an equivalence relation, but it is harder to determine the equivalence classes. Later we show there is a more practical way of describing this relation that makes the computation of the equivalence classes easier.

The reflexive and symmetric properties of $\sim$ are clear, and the transitive property requires only a little attention to the proof of Theorem 2, which can be supplied by the reader.

The graph in Figure 1a provides a simple illustration of this equivalence relation. The two edges of weight three on the left are equivalent to each other but not to either of the edges of weight three on the right.

Since every equal edge replacement preserves equivalence classes, we obtain two refinements of Theorem 3:

THEOREM 7. *All minimum spanning trees contain the same number of edges from any given equivalence class.*

THEOREM 8. *An edge is in every minimum spanning tree of G if and only if it is related only to itself.*

We now give an elementary algorithm for finding the equivalence classes, using a reference tree. It is shown in **[6]** that the equivalence relation produced by this algorithm is the same as the one defined earlier.

Recall that the edges that do not belong to any minimum spanning tree have been deleted from G. For each remaining edge g not in the reference tree $\mathscr{T}$, let $X(g)$ be the set consisting of g and all edges on the path in $\mathscr{T}$ joining the vertices of g whose weight is equal to $w(g)$. This set is illustrated in Figure 2, where $X(g) = \{g,e,f\}$.

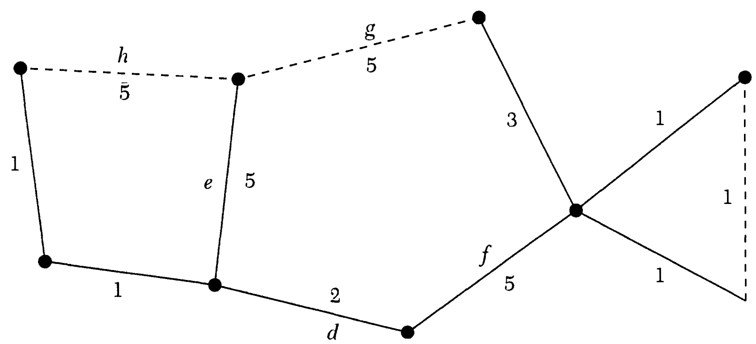

FIGURE 2

Combine each pair of sets $X(g)$ that have an edge in common. Each resulting combined set forms one equivalence class. Thus, each such equivalence class is a transitivity class of the sets $X(g)$. Edges of G that belong to none of the $X(g)$ form singleton equivalence classes.

In Figure 2, $X(g)$ and $X(h)$ have the edge e in common and are combined. Since the edge d, for example, belongs to none of the sets $X(c)$, $X(g)$, $X(h)$, we see that $[d] = \{d\}$ and d must belong to every minimum spanning tree.

Since every minimum spanning tree of G must contain the same number of edges from a given equivalence class $[e]$, it is reasonable to ask whether any subset of $[e]$ with that same number of edges can be found in some minimum spanning tree. The answer is negative, and a simple example can be found in K_4, the complete graph on four vertices, in which all six edges have been assigned the same weight. All the edges are equivalent to each other and every minimum spanning tree has three edges, but some 3-element subsets are clearly not trees.

In general, we say that a subset S of $[e]$ is a *choice* from $[e]$ if S is the set of edges in $[e]$ in some minimum spanning tree. The main result in **[6]**, which is not obvious, is as follows:

THEOREM 9. *If S_1 and S_2 are two choices from $[e]$ and T is any minimum spanning tree of G containing S_1, then $(T - S_1) \cup S_2$ is also a minimum spanning tree.*

This theorem guarantees that the choices from a given class are fully interchangeable, and that the set of all minimum spanning trees in G can be generated algorithmically by taking one choice from each equivalence class. In particular, the number of minimum spanning trees of G is the product of the numbers of choices from all the classes.

Figure 3 offers an opportunity to execute this algorithm. In Figure 3a, a weighted graph is given. Each edge has a letter (its name) and a number (its weight). In Figure 3b, a reference tree $\mathscr{T}$ (dark lines) has been chosen arbitrarily. Three edges have been rejected because they do not belong to any minimum spanning tree (Theorem 6), and three others are shown with a dashed line because they do not belong to the reference tree. For the three dashed edges c, h and l, find $X(c)$, $X(h)$ and $X(l)$ and observe that $X(h)$ and $X(l)$ can be combined.

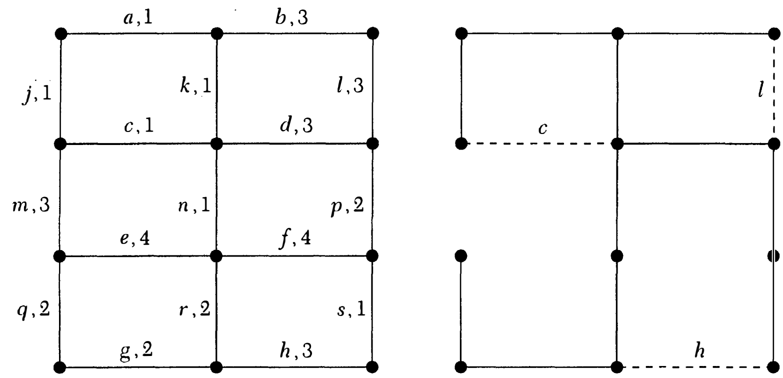

FIGURE 3

The equivalence classes are $\{a, c, j, k\}$, $\{b, d, h, l\}$, and six other singleton classes. The reader can verify that $\{a, c, j, k\}$ has four choices, each of which has three edges, and that $\{b, d, h, l\}$ has five choices, each with two edges. Each singleton class has, of course, only one choice.

Since there are twenty ways to select one choice from each class, G has exactly twenty minimum spanning trees.

For small graphs, this process can be done visually. For larger graphs, determinants can be used to identify the choices. In any event, Theorem 9 greatly increases the efficiency of the search. Once the equivalence classes have been found, we can determine the choices from any class $[e]$ by simply replacing the choice from $[e]$ in the reference tree $\mathscr{T}$ by another set C of edges from $[e]$ with the same cardinality and computing the determinant of the new submatrix of A so obtained. If the determinant is zero, the set C is not a choice. If the determinant is non-zero, it will have $\pm \det M(\mathscr{T})$ as its value, and thus the set C is a choice. Note that by proceeding in this manner, we never waste time evaluating the determinant of a non-minimum spanning tree.

The number of determinants to be evaluated is thus reduced from $\binom{b}{n-1}$ to

$$\binom{t_1}{i_1} + \binom{t_2}{i_2} + \cdots + \binom{t_m}{i_m},$$

where t_j is the number of edges in the j^{th} equivalence class E_j, and i_j is the number of edges from E_j that belong to the reference tree.

It is hard to make a general statement about the complexity of this algorithm, because it depends heavily on the number of equivalence classes and the distribution of the edges of G among them. Observe that the complexity decreases as the number of equivalence classes increases.

The graph in Figure 3 has 12 vertices and 14 usable edges. Thus $\binom{b}{n-1} = \binom{14}{11} = 364$ determinants would have to be computed to find the 20 minimum spanning trees. If the algorithm is used, only

$$16 = \binom{4}{2} + \binom{4}{3} + \binom{1}{1} + \binom{1}{1} + \binom{1}{1} + \binom{1}{1} + \binom{1}{1} + \binom{1}{1}$$

determinants need to be computed to determine the choices from all the equivalence classes. This is overstated, since the reference tree has already identified one choice from each class, and so only 8 determinants need to be computed to identify the remaining choices. This improvement must be weighed against the initial effort of finding a reference tree and computing the equivalence classes.

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What is the calculus?

Most people have heard only of "the calculus," a slip-shod expression applied to the infinitesimal calculus of mathematics invented by Leibniz and Newton. That calculus is undoubtedly the most impressive and important ever constructed, and we may forgive its admirers for trying to pre-empt the word "calculus" as its proper name; yet this word is too useful to be lost from the more general science of logic. A calculus is, in fact, any system wherein we may calculate. Ordinary arithmetic, the system of natural numbers with its constituent operations $+$, $\times$, $\div$, and $-$, is a calculus; the famous "hedonistic calculus" of Jeremy Bentham was so named in the fond and sanguine belief that this philosophy furnished a system wherein the relative magnitudes of the pleasures could be exactly calculated. But as it involved no operations upon the elements called "pleasures," it failed to be a calculus.

From Susanne K. Langer's *Introduction to Symbolic Logic*, published 1937; quoted in the February 1940 issue of the *National Mathematics Magazine* (this *Magazine*'s predecessor).

NOTES

Dinner, Dancing, and Tennis, Anyone?

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Introduction In August 1996 the United States Tennis Association (USTA) had a problem. They had just conducted the men's draw for the annual U.S. Open tournament, but, contrary to previous years' procedure, the draw was made before the seedings were determined for the top 16 of the 128 players. Worse yet, the seedings announced did not follow the standard computer rankings for the top players, as had long been the policy. Americans Michael Chang and André Agassi, for example, were ranked third and eighth, but were seeded second and sixth, respectively. Amid allegations that the seedings were rigged to promote late-round match-ups between high-visibility players, the USTA conceded that this unusual procedure had created at least the appearance of impropriety, and agreed to redo the draw, this time after the seedings had been announced.

This unprecedented re-draw did not satisfy all of the critics, particularly those who felt that the seedings favored Americans, but it did quiet talk of a boycott from several players. It also raised some interesting questions about probability. In the first draw, Andrei Medvedev of the Ukraine was pitted against Jean-Philippe Fleurian of France. In the re-draw, these same two unseeded players were scheduled to meet in the first round. A USTA official found this a remarkable coincidence, and contacted the Department of Mathematics and Computer Science at nearby St. John's University to inquire about the probability of such an event.

I happened to field the phone call. It took some time to determine just what the official was asking. For example, consider the difference between the following two questions:

> *The Easy U.S. Open Question*: What is the probability that Medvedev and Fleurian would be drawn as first-round opponents in both draws?
>
> *The Hard U.S. Open Question*: What is the probability that at least one pair of players would be drawn as first-round opponents in both draws?

The official seemed to be more interested in the first question, whose answer is much smaller and much easier to obtain than the answer to the second question. The second question turned out to be subtler than expected. It also brought to mind several related questions that were also interesting in their own right—two problems that I refer to as the dinner problem and the dancing problem. The dinner problem is a generalization of the classical problem of coincidences, first discussed by Montmort in 1708. The dancing problem is related to another classical problem, the *problème des ménages*, in which n married couples are seated at a round table. Both problems can be defined recursively, and solutions are readily obtained for finite values by direct computation. Solutions to the dinner and dancing problems will be used to answer the Hard U.S. Open Problem.

The dinner problem I first give the simple version of the problem, and then throw in a twist.

> *The Dinner Problem (simple version)*: Suppose n people are invited to a dinner party. Seats are assigned and a name card made for each guest. However, floral arrangements on the tables unexpectedly obscure the name cards. When the n guests arrive, they seat themselves randomly. What is the probability that no guest sits in his or her assigned seat?

Another equivalent formulation of this problem is the well known hat-check problem, in which a careless attendant loses all of the slips and returns the hats at random. The probability that nobody gets his or her own hat is equal to the probability in the simple version of the dinner problem.

We will generalize the dinner problem, for two reasons. First, we will need the answer to the more general problem to answer the Hard U.S. Open question. Second, and perhaps surprisingly, the recursion formulas are more easily developed with the more general problem.

> *The Dinner Problem (with party crashers)*: Suppose n people are invited to a dinner party as before, with the same confusion about the seating arrangement. This time k of the n diners are party crashers, where $0 \le k \le n$. (No name cards exist, of course, for the party crashers.) Once again, when the n diners arrive, they seat themselves randomly at the tables. What is the probability $p_{n,k}$ that no invited guest sits in his or her assigned seat?

The simple version of the problem, with no party crashers, asks for $p_{n,0}$.

Recursive formulas for the dinner problem We give the first few cases for small values of n and k, and derive the recursive equations. If there is one guest ($n = 1$), then she is either invited ($k = 0$), or not ($k = 1$). In the first case she must sit in her own seat and $p_{1,0} = 0$; otherwise, she cannot sit in her own seat, and $p_{1,1} = 1$.

If $n > 1$, we can express the probability $p_{n,k}$ in terms of probabilities involving one fewer guests, in one of two ways. If k is positive, then there is at least one party crasher. Ironically, it is easier to establish an equation in this case if we abandon all pretense towards social fairness and seat a party crasher first. There is no possibility that the party crasher will sit in her assigned seat; the probability is $\frac{k}{n}$ that she will sit at a place designated for one of the k absent invitees. If she sits in one of these k seats, then the probability that none of the remaining $n - 1$ guests will sit in his or her assigned seat is, by definition, $p_{n-1,k-1}$. On the other hand, the probability that she will sit in one of the $n - k$ seats assigned to a guest who is present is $\frac{n-k}{n}$. In this event, then $n - 1$ guests will remain to be seated, of whom $k - 1$ are party crashers and one was invited—but just had his seat taken by the party crasher who was seated first! This invitee no longer has any chance of sitting in his own seat, and so becomes indistinguishable from the remaining $k - 1$ party crashers. Therefore, the probability that none of the remaining $n - 1$ guests will sit in his or her assigned seat is $p_{n-1,k}$. In summary we have the following reduction formula for $n > 1$ and $k > 0$:

$$p_{n,k} = \frac{k}{n} p_{n-1,k-1} + \frac{n-k}{n} p_{n-1,k} \tag{1}$$

If there are no party crashers ($k=0$), we seat one invited guest at random. The probability is $\frac{n-1}{n}$ that she will not sit in her own seat, in which case one of the remaining $n-1$ guests can no longer sit in his own seat. For the problem of seating the remaining $n-1$ guests, this displaced soul becomes, in essence, a party crasher himself. The probability that none of the remaining $n-1$ guests will sit in his or her assigned seat is therefore $p_{n-1,1}$. Thus, the reduction equation for the case where there are no party crashers is

$$p_{n,0} = \frac{n-1}{n} p_{n-1,1}. \tag{2}$$

Equations 1 and 2, together with the base cases described above, allow us to generate as many of the probability numbers, where $0 \le k \le n$, as patience or computer speed will allow. Table 1 gives values for $n \le 10$.

TABLE 1. The Dinner Problem Probabilities $p_{n,k}$, $0 \le k \le n \le 10$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1.0000										
1	0.0000	1.0000									
2	0.5000	0.5000	1.0000								
3	0.3333	0.5000	0.6667	1.0000							
4	0.3750	0.4583	0.5833	0.7500	1.0000						
5	0.3667	0.4417	0.5333	0.6500	0.8000	1.0000					
6	0.3681	0.4292	0.5028	0.5917	0.7000	0.8333	1.0000				
7	0.3679	0.4204	0.4817	0.5536	0.6381	0.7381	0.8571	1.0000			
8	0.3679	0.4139	0.4664	0.5266	0.5958	0.6756	0.7679	0.8750	1.0000		
9	0.3679	0.4088	0.4547	0.5066	0.5651	0.6313	0.7063	0.7917	0.8889	1.0000	
10	0.3679	0.4047	0.4455	0.4910	0.5417	0.5982	0.6613	0.7319	0.8111	0.9000	1.0000

Notice the rapid convergence of the numbers $p_{n,0}$ in the first column. Indeed, there is a well-known non-recursive formula for these simple dinner problem probabilities (see **[2]**, **[4]**, or **[7]**):

$$p_{n,0} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

so that the limit is $e^{-1} \approx 0.3679$.

The dancing problem In order to solve our Hard U.S. Open Question, we need to double the fun we have had with the dinner problem and consider the dancing problem. Again, there will be both a simple version and one with a twist, but this time we will have twice as many people to work with.

> *The Dancing Problem* (*simple version*): Suppose n married couples ($2n$ people) are invited to a party. Dance partners are chosen at random, without regard to gender. What is the probability that nobody will be paired with his or her spouse?

Just as dance partners are paired without regard to gender, we do not assume, as current laws in most states do, that a married couple must consist of a male and a female.

Again, we answer a simple question by posing a harder, more general one. This time, we introduce single guests into the problem.

The Dancing Problem (with single people): Suppose $n-k$ married couples and $2k$ single people are invited to a party. (That's still $2n$ people.) Dance partners are chosen at random, without regard to gender. What is the probability $d_{n,k}$ that nobody will be paired with his or her spouse?

The simple version of the problem, with no single people, is to determine $d_{n,0}$.

Recursive formulas for the dancing problem We consider the base cases first. If there are no guests, then no spouses can be paired, so $d_{0,0} = 1$. If there are only two guests ($n = 1$), then either they are married, in which case $d_{1,0} = 0$; or they are both single, in which case $d_{1,1} = 1$.

To obtain recursive formulas for the dancing problem, we assume that there are at least two people invited ($n \geq 1$), and then express $d_{n,k}$ in terms of probabilities with strictly smaller values of n. If there are any single people ($k > 0$), we pair one of them first. We select a single person, whom we'll call Sam. Of the remaining $2n - 1$ people, $2k - 1$ are single and $2n - 2k$ are married. The probability that Sam will be paired with another single person is $\frac{2k-1}{2n-1}$. In that case, the probability of pairing off the remaining $2n - 2$ people, of whom $2k - 2$ are single, so that no spouses are paired together is $d_{n-1,\,k-1}$. On the other hand, the probability that Sam will be paired with a married person is $\frac{2n-2k}{2n-1}$. In this case, the remaining $2n - 2$ people comprise $2k - 1$ single people, $n - k - 1$ married couples, and one leftover married person whose spouse was paired with Sam. For the purposes of pairing off the remaining $2n - 2$ people into dance partners, this leftover spouse has become indistinguishable from the other single people, since he or she cannot be paired with his or her spouse. Sam has, in effect, broken up this marriage! Thus, the probability that none of the remaining $2n - 2$ people will be paired with a spouse is $d_{n-k,k}$, since there are still $2k$ effectively single people. Putting this together gives the following reduction formula, provided $n > 1$ and $k > 0$:

$$d_{n,k} = \frac{2k-1}{2n-1} d_{n-1,k-1} + \frac{2n-2k}{2n-1} d_{n-1,k}. \tag{3}$$

If there are no single people ($k = 0$), then we must pair a married person first. The probability that this married person will not be paired with his or her spouse is $\frac{2n-1}{2n-2}$. In this case, that leaves $2n - 2$ people to be paired as dance partners, including two lone people whose spouses were just paired together. Since neither of these people can be paired with a spouse, they are considered single for the problem of pairing the remaining $2n - 2$ people. The probability that none of the remaining $2n - 2$ people will be paired with a spouse is thus $d_{n-1,\,1}$, yielding the following formula, when $n > 1$:

$$d_{n,0} = \frac{2n-2}{2n-1} d_{n-1,1}. \tag{4}$$

Once again, we can generate all desired values of $d_{n,k}$, for $0 \leq k \leq n$, using equations (3) and (4), and the base cases derived above. The values for $0 \leq k \leq n \leq 10$ and several values of $d_{n,0}$ for higher values of n appear in Table 2.

Notice that the sequence of numbers in the first column of Table 2, corresponding to the simple version of the dancing problem, does not converge nearly as rapidly as in the simple dinner problem. Nonetheless, it does seem to converge to a number strictly less than 1. Can you guess the limit?

TABLE 2 The Dancing Problem Probabilities $d_{n,k}$, $0 \le k \le n \le 10$

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1.0000										
1	0.0000	1.0000									
2	0.6667	0.6667	1.0000								
3	0.5333	0.6667	0.8000	1.0000							
4	0.5714	0.6476	0.7429	0.8571	1.0000						
5	0.5757	0.6392	0.7111	0.7937	0.8889	1.0000					
6	0.5810	0.6334	0.6915	0.7561	0.8283	0.9091	1.0000				
7	0.5847	0.6294	0.6781	0.7313	0.7894	0.8531	0.9231	1.0000			
8	0.5874	0.6264	0.6683	0.7135	0.7623	0.8149	0.8718	0.9333	1.0000		
9	0.5895	0.6241	0.6609	0.7002	0.7422	0.7871	0.8350	0.8863	0.9412	1.0000	
10	0.5912	0.6223	0.6551	0.6899	0.7268	0.7658	0.8072	0.8512	0.8978	0.9474	1.0000
100	0.6050124905										
1000	0.6063790072										
10,000	0.6065154929										
100,000	0.6065291359										

Why the Hard U.S. Open Question is hard We now have almost all of the necessary tools to answer the Hard U.S. Open Question. We recall that in each draw there are 128 players, of whom 16 are seeded; seeded players cannot meet in the first round.

If there were no seedings at all, then we could answer this question directly using the dancing numbers $d_{n,k}$. Imagine pairing up the 128 players into 64 married couples according to the first-round match-ups in the first draw. Then we pair up dance partners according to the first-round match-ups in the second draw. Having at least one pair of players drawn as first-round opponents in both draws would be equivalent to having at least one married couple paired as dance partners. This is the complement of having no spouses paired as dance partners, so the answer to the Hard U.S. Open Problem, *if there were no seedings*, would be

$$1 - d_{64,0} \approx 1 - 0.604157 = 0.395843.$$

But then it wouldn't be a very hard problem, would it?

In addition to the dinner and dancing probability numbers $p_{n,k}$ and $d_{n,k}$, we will use a simpler, non-recursive combinatorial formula. Suppose a sample of m people is picked at random from a group of n couples. (That's $2n$ people, with no singles.) We denote by $r(n, m, k)$ the probability that exactly k of the n couples will be included in the sample of m people, where $0 \le k \le \lfloor m/2 \rfloor$.

We leave as an exercise to the interested reader to show that

$$r(n,m,k) = \frac{\binom{n}{k}\binom{n-k}{m-2k}2^{m-2k}}{\binom{2n}{m}}.$$

Answering the Hard U.S. Open Question The principal effect of the seedings on our Question is that no two seeded players can meet as first-round opponents. Notice that there are two different ways in which players might be paired as first-round opponents in both draws. An unseeded player might draw the same seeded opponent twice, or he might draw the same unseeded opponent twice. We use the dinner problem probabilities to address the first issue, and the dancing number probabilities for the second. Of the 128 players, 16 are seeded and 112 are unseeded. Of the 112 unseeded players, 96 are initially lucky because they drew unseeded opponents in the first draw. The answer to the Hard U.S. Open Question will depend on how many of these 96 "lucky" unseeded players happen to draw a seeded opponent in the second draw. We let m denote the number of unseeded players who drew an unseeded opponent in the first draw, but a seeded opponent in the second draw. These players were thus rather unlucky, and most likely none too pleased with the decision to remake the draw.

This integer m can range in value from 0 to 16. Let E_m denote the event described above, that exactly m unseeded players drew an unseeded opponent in the first draw, but a seeded opponent in the second draw. The probability that the event E_m occurs is given by

$$P(E_m) = \frac{\binom{96}{m}\binom{16}{16-m}}{\binom{112}{16}} = \frac{\binom{96}{m}\binom{16}{m}}{\binom{112}{16}}.$$

The denominator counts all possible ways of picking 16 unseeded players to play against seeded players in the second draw. In the numerator, of the 96 unseeded players who drew unseeded opponents in the first draw, we choose m of them to have seeded opponents in the second draw. Then, of the 16 unseeded players who drew seeded opponents in the first draw, we choose $16-m$ of them to have seeded opponents again in the second draw. (These are the doubly unlucky unseeded players!)

Now, suppose that the event E_m occurs. The probability that, of the $16-m$ players who drew seeded opponents twice, none of them drew the same seeded opponent in both draws is exactly the dinner problem probability number $p_{16,m}$. To see this, think of the seeded players as the place settings in the dinner problem. The 16 unseeded players who drew seeded opponents in the first draw are the invited guests, whose (hidden) name cards are at the place settings corresponding to their seeded opponents in the first draw. The m unseeded players who drew unseeded opponents in the first draw, and seeded opponents in the second, are the m party crashers, since they cannot be matched with the same opponent twice. The $16-m$ players who drew seeded opponents in both draws are the invited guests that actually made it to the dinner party.

Now we consider the possibility that two unseeded players were chosen as first-round opponents in both draws. Continue supposing that event E_m occurs. Of the m players who drew an unseeded opponent in the first draw and a seeded opponent in the second draw, suppose that k pairs of them were scheduled against each other as opponents in the first draw. The probability of this event is given by the number

$$r(48, m, k) = \frac{\binom{48}{k}\binom{48-k}{m-2k}2^{m-2k}}{\binom{96}{m}}$$

discussed earlier, where $0 \le k \le \lfloor m/2 \rfloor$.

To summarize, we are supposing that event E_m occurs; that is, that there are m unseeded players who drew an unseeded opponent in the first draw and a seeded opponent in the second draw; and that of those m players, k pairs of them were scheduled as opponents in the first draw. Given those conditions, what is the probability that, of the $96-m$ unseeded players who drew unseeded opponents in both draws, no two were scheduled against each other in both draws? The answer, of course, is the dancing problem probability number $d_{48,m-k}$.

How is that? Marital status is determined by how the players were matched up in the first draw, while dancing partners are chosen by the match-ups in the second draw. There are 96 unseeded players who drew unseeded opponents in the first draw. These 96 people represent 48 originally married couples in the dancing problem, paired by their scheduled opponents in the first draw. Of these 96 players, m drew seeded opponents in the second draw. These m players represent some, but not all, of the single guests, since they cannot be matched with their "spouses" in the second draw. Now, since we are assuming that k pairs of those m players were paired up in the first draw, $m-2k$ of these m players' first-draw opponents (spouses) must have drawn unseeded opponents in the second draw. These $m-2k$ spouses are effectively single as well, since they cannot be matched with their first-draw opponents in the second draw. This gives a total of $m+(m-2k)=2(m-k)$ single people out of $2\cdot 48=96$ guests. The probability that no two of the 96 players will be paired together in both draws is thus $d_{48,\,m-k}$.

The answer to the Hard U.S. Open Question To answer the Hard U.S. Open Question at last, we add up the probabilities for each of the possible values of k, $0\le k\le \lfloor m/2\rfloor$, for each of the possible values of m, $0\le m\le 16$. Thus, the probability that no two players would be drawn as first round opponents in both draws is

$$\sum_{m=0}^{16} P(E_m)\,p_{16,\,m}\left(\sum_{k=0}^{\lfloor m/2\rfloor} r(48,m,k)\,d_{48,\,m-k}\right)\approx 0.5983933573.$$

A computer algebra system can give the exact answer in rational form. Curiously, in this case it is a (reduced) fraction of integers, each with exactly 100 digits.

The probability that at least one pair of players would be drawn as first-round opponents in both draws is, accurate to ten decimal places, $1-0.5983933573=0.4016066427$.

Recall that if there were no seedings the answer would have been $1-d_{64,\,0}\approx 0.395843$, which is fairly close to the actual answer.

How does our answer to the Hard U.S. Open Question compare to that of the Easy U.S. Open Question: "What is the probability that Medvedev and Fleurian would be drawn as first-round opponents in both draws?" We leave it to the interested reader to show that the answer is

$$\left(\frac{2}{259}\right)^2=\frac{4}{67{,}081}=\frac{1}{16{,}770.25}\approx 0.00005963.$$

Conclusions It is indeed quite improbable that any two particular players would be drawn as first-round opponents in both draws. This is in dramatic contrast to the answer to the Hard U.S. Open Question, where we saw that there is a greater than 40% chance that at least one pair of players will be drawn as first-round opponents in both draws. What I find most surprising is not the disparity in the two answers, but the subtlety required to answer the Hard U.S. Open Question. Along the way, we explored two related problems and developed formulas for each of them. It is remarkable that in the simple versions of both of these problems, the probabilities do not converge to either of the extreme values, 0 or 1, as the number of guests increases without bound. In the simple version of the dinner problem, the probability approaches $e^{-1}\approx 0.3679$, and the convergence is quite rapid. In the simple version of the dancing problem, we conjecture that the probability converges to $\frac{1}{\sqrt{e}} \approx$ 0.606530659. Finally, we leave the interested reader with a few problems to explore.

1. Determine the probability that *exactly* k pairs of players will be selected as first-round opponents in both draws, for $0 \le k \le 64$; then calculate the expected number of repeated pairs.
2. Find a non-recursive formula for the simple dancing problem probabilities, $d_{n,0}$.
3. Prove (or disprove) that $\lim_{n\to\infty} d_{n,0} = \frac{1}{\sqrt{e}}$.

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Variations on a Theme: A_4 Definitely Has No Subgroup of Order Six!

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Introduction To obtain *one* valid proof of a theorem is an achievement, but there may be many different proofs of the same theorem. For example, there are said to be over 370 of Pythagoras's theorem. Once a result has been proved, the story seldom ends. Instead the search begins for refined, reduced, or simplified proofs.

It is just as important to have a collection of different approaches to proving a given result as it is to have a collection of different results that can be derived using a given technique. An advantage of this attitude is that if one has already proved a result using a certain technique, then a different method of proving the same result may sometimes yield a generalization of the original result which may *not* be possible with the original technique of proof. We illustrate this phenomenon by examining various proofs of the fact that A_4, the alternating group on four symbols, has no subgroup of order six.

Preliminaries One of the cornerstones of theory of finite groups is the following theorem of the Italian mathematician J. L. Lagrange (1736–1813):

LAGRANGE'S THEOREM. *If G is a finite group with $|G| = n$ and H is a subgroup of G with $|H| = d$, then d is a divisor of n.*

Lagrange stated the theorem for the special case where G was a subgroup of the symmetric group S_n which arose out of his study of the permutations of the roots of a polynomial equation. The theorem as stated above was probably first proved by Galois **[9]** around 1830.

Is the converse true?

CONVERSE TO LAGRANGE'S THEOREM. *If G is a finite group with $|G| = n$, and d is a divisor of n, then G has a subgroup of order d.*

It is well known that this converse is false, and that a counterexample of smallest order is provided by A_4, the alternating group on 4 symbols. This group of order 12 has no subgroup of order 6. We write A_4 as the group of all even permutations on the four symbols $\{1, 2, 3, 4\}$.

$$A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (132), \\ (124), (142), (134), (143), (234), (243)\}$$

We now present eleven elementary proofs of the fact that A_4 has no subgroup of order 6. Several attempts **[2, 4, 6]** at presenting the "simplest" or "best" proof of showing that A_4 has no subgroup of index 2 have recently been made. Since notions like "best" or "simplest" proof are subjective, we present a range of possible candidates. All eleven proofs involve only elementary concepts from group theory: cosets, element orders, conjugacy classes, normality, isomorphism classes, commutator subgroup, cycle structure. The variety of topics that arise is a valuable review of basic group theory!

Proofs of the falsity of the converse Let H be an alleged subgroup of A_4 of order 6. Each proof following implies that such an H cannot exist. Of course, the most simple-minded approach is to look at all $\binom{12}{6} = 924$ subsets of A_4 and show that none of them forms a subgroup. However, as *Proof* 1 illustrates, this number can be halved immediately.

Proof 1. (Basic but crude)

H must contain the identity element e, so H has five nonidentity elements. There are $\binom{11}{5} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 462$ possible subsets to consider. We leave it for the reader to check that none of these 462 subsets is closed under composition of cycles. This is an arduous task to undertake by hand but quite feasible for a computer where the Cayley table for A_4 has been entered. Paradoxically, this crude approach forms the basis of a later proof which we nominate as the "simplest" but not "easiest" proof of the converse.

Proof 2. (Using cosets)

Since H has index 2 we have $A_4 = H \cup Ha$ for all $a \in A_4 \setminus H$. Consider Ha^2; now $Ha^2 = H$ or $Ha^2 = Ha$. If $Ha^2 = Ha$, then $Ha = H$ by cancellation and $a \in H$, a contradiction. Thus $Ha^2 = H$; but $Hh^2 = H$ for all $h \in H$ and so $Hg^2 = H$ for all $g \in A_4$. Thus $g^2 \in H$ for all $g \in A_4$. By direct calculation A_4 has nine distinct squares, so $|H| \geq 9$, contradicting $|H| = 6$.

Proof 3. **[4]** (A variation of Proof 2)

As in Proof 2 we have $g^2 \in H$ for all $g \in A_4$. If $a^3 = e$ then $a^2 = a^{-1}$, so $a^2 \in H \Rightarrow a^{-1} \in H \Rightarrow a \in H$. But this would mean that H contains all eight elements of order 3 in A_4, which is a contradiction.

Proof 4. (Using normality)

A subgroup of index 2 is also a normal subgroup. Hence $H \lhd A_4$ and the factor group A_4/H is a cyclic group of order 2. Thus $H = (Hg)^2 = Hg^2$ for all $g \in A_4$, so $g^2 \in H$. We finish the proof using the same argument as in Proof 2.

Proofs 2, 3, and 4 display the characteristic that was mentioned in the introduction; they generalize easily to yield the following result.

THEOREM. *Let G be a finite group of even order and suppose that more than half the elements of G have odd order. Then G has no subgroup H of index* 2.

This result implies that the direct product $A_4 \times C_n$, where C_n is the cyclic group of odd order, has no subgroup of index 2. Thus there exists a counterexample to the converse of Lagrange's theorem of order $12n$ for each odd integer n.

Proof 5. (Using conjugacy classes)

The conjugacy classes of A_4 are

$$\{e\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (124), (134), (234)\},$$
$$\{(132), (142), (143), (243)\}$$

with cardinalities 1, 3, 4, and 4 respectively. Since H has index 2, H is a normal subgroup of A_4 and so H must consist of *complete* conjugacy classes, one of which must be $\{e\}$. But it is clearly not possible to make up the 5 remaining elements with sets of size 3 and 4. Hence H does not exist.

Proof 6. (Using isomorphism classes)

Since $|H| = 6$, H must be isomorphic to one of the following groups; S_3, the group of all permutations on 3 symbols $\{a, b, c\}$ or C_6 the cyclic group of order 6. Since A_4 clearly has no element of order 6 the latter possibility is ruled out. Hence $H \approx S_3$. Now S_3 has exactly three elements of order 2, namely $X = \{(ab), (bc), (ac)\}$ and A_4 (and hence H) has exactly three elements of order 2, given by $Y = \{(12)(34), (13)(24), (14)(23)\}$. The isomorphism, which preserves the order of an element, must map Y onto X. But the elements of Y commute pairwise whereas no two distinct elements of X commute. This contradicts a property of isomorphisms and hence these groups cannot be isomorphic. We conclude that H does not exist.

Proof 7. (Variation on Proof 6)

$H \approx S_3$ implies that H contains the three elements of A_4 of order 2, and therefore H contains $V = \{e, (12)(34), (13)(24), (14)(23)\}$. But V is a group of order 4 and 4 does not divide 6, contradicting Lagrange's theorem.

Proof 8. (Using the commutator subgroup)

Since H is a subgroup of index 2, $H \lhd A_4$ and the factor group A_4/H is an abelian group of order 2. Thus $H \supseteq A_4'$ where A_4' denotes the commutator subgroup. A little computation shows that $A_4' = \{e, (12)(34), (13)(24), (14)(23)\}$. As in Proof 7, 4 does not divide 6, again contradicting Lagrange's theorem.

Proof 8 offers an easy alternative proof of the result of Mackiw **[9]** that the group $SL(2,3)$ (the group of all 2×2 invertible matrices of determinant 1 with entries in $\mathbf{Z}_3$)

of order 24 has no subgroup of order 12. If K is such a subgroup, then $K \triangleleft SL(2,3)$ and since the factor group $SL(2,3)/K$ is abelian, $K \supseteq SL(2,3)'$, the commutator subgroup. But it is easy to see that $|SL(2,3)'| = 8$ and we get a contradiction since 8 does not divide 12.

Proof 9. (Using normal subgroups)

The group $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4 as is the subgroup H. Since HV contains V properly and V is a maximal subgroup we must have $HV = A_4$. By a well known result **[9]**,

$$12 = |A_4| = |HV| = \frac{|H||V|}{|H \cap V|} = 12 = \frac{6 \times 4}{|H \cap V|}.$$

Hence $|H \cap V| = 2$. But this is a contradiction since A_4 has no *normal* subgroup of order 2, as is easily checked.

In the final two proofs all that is used is the closure property, i.e., if $a, b \in H$ then $ab \in H$.

Proof 10. **[1]** (Using order of an element)

Since $e \in H$ there is space only for five remaining elements in H. The elements of A_4 are either of order 2 or of order 3. Elements of order 3 occur in pairs and hence we must have an *even* number of elements of order 3 in H. Since A_4 has eight elements of order 3 and only three elements of order 2, H must contain at least one element of order 3, and, because elements of order 3 come in pairs (ρ and ρ^2), there are two possible cases to consider.

Case I. H contains four distinct elements of order 3, say ρ, ω, ρ^2, ω^2.

In addition to the above four elements and the identity we would also get the distinct elements $\rho\omega$ and $\rho\omega^2$. Note that $\rho\omega \neq \rho, \omega, \rho^2$ or ω^2 since otherwise, by cancellation we get that $\omega = e$, or $\rho = e$ or $\rho = \omega$, all of which are false. Similarly the element $\rho\omega^2$ is distinct from the six elements $e, \rho, \omega, \rho^2, \omega^2, \rho\omega$. Hence $|H| \geq 7$, a contradiction.

Case II. H contains exactly two elements of order 3, say ρ, ρ^2

This would mean that H contains e and the 3 elements of order 2, which form a subgroup of order 4, contradicting Lagrange's theorem.

We contend that the final proof is possibly the most elementary of all the proofs in that it utilizes only the closure property. It does involve a bit of computation but the number of cases to check is far more manageable than in Proof 1.

Proof 11. Partition G into "packets" as follows $\{(e)\}$, $\{(12)(34)\}$, $\{(13)(24)\}$ $\{(14)(23)\}$, $\{(123), (132)\}$,$\{(124), (142)\}$, $\{(134), (143)\}$, $\{(234), (243)\}$. Note that by closure, H must contain all the elements of a packet or no element of a packet.

Now $e \in H$ so H is made up of either

(i) three 1-packets and one 2-packet and e; or
(ii) one 1-packets and two 2-packets and e.

This gives $\binom{3}{3} \cdot \binom{4}{1} + \binom{3}{1} \cdot \binom{4}{2} = 1 \cdot 4 + 3 \cdot 6 = 22$ sets to be checked for closure. In each of the 22 sets, elements a and b can be found such that $ab \notin H$. Hence no such H exists.

We remark that in several textbooks **[3, 5, 7]**, the problem of disproving the converse to Lagrange's theorem is often relegated to an exercise. Sadly sometimes the proof is dismissed with the words "It can be shown," "As one can easily see," "It will be found." Other texts offer proofs that involve complicated arguments **[10]**. We invite readers to add to the above list of elementary proofs or variations of proofs.

Acknowledgment. We wish to thank the referee, whose suggestions led to a considerable improvement in the presentation of this paper.

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A Principle of Countability

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Introductory courses in undergraduate analysis usually include a proof of the fact that the rational numbers are countable. In a note appearing in 1986 [**1**], Campbell presents an alternative to the usual diagonalization argument. Touhey's proof in the 1996 article [**4**] proceeds along similar lines. In both of these papers, two sets are declared to have the same cardinality if each can be mapped in a one-to-one manner into the other. Most sources refer to this condition as the Cantor–Bernstein Theorem [**5**, p. 103] or the Schröder–Bernstein Theorem [**2**, p. 99; **3**, p. 74], a deeper result that may not appear in an introductory analysis course.

In this note, I state a principle of countability and illustrate how it may be applied both to argue the countability of some familiar sets and to prove two well-known general results about countable sets. The main difference between the present approach and that in [**1**] and [**4**] is that, here, countability is established without any mention of Cantor/Schröder–Bernstein, but rather by appealing to the definition of, and an elementary result about, countable sets. The function defined in establishing the principle here is also slightly more general. The principle is likely part of the lore of the subject of infinite sets, but it certainly deserves to be better known. It appears in no textbook from which I have studied or taught. The underlying idea was shown to me in graduate school by Professor John L. Troutman at Syracuse University.

A set S is called *finite* if, for some natural number n, there is a one-to-one, onto function between S and the initial segment $\{1, 2, \dots, n\}$ of the set of natural numbers $\mathbb{N}$. If there is a one-to-one, onto function between S and the set $\mathbb{N}$, then S is called *countably infinite*. A set that is either finite or countably infinite is said to be *countable*.

The main ingredients of the result that follows are a fixed, finite base set, called the *alphabet*, the elements of which are called *letters*, and the *words* that may be formed

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The main ingredients of the result that follows are a fixed, finite base set, called the *alphabet*, the elements of which are called *letters*, and the *words* that may be formed using the letters of the alphabet. A word is formed by juxtaposing letters of the alphabet to form a formal string of finite length. (Words may have repeated letters.) With this vocabulary in place, the principle is as follows:

PRINCIPLE OF COUNTABILITY. *The set of all words that may be formed using letters of a finite alphabet is countably infinite. Hence, any set of words that may be so formed is countable.*

To sketch a proof of this principle, suppose the alphabet A consists of r letters, $r \in \mathbb{N}$. For each $n \in \mathbb{N}$, let E_n be the set of all words of length n that are obtainable using letters from A. Then E_n is a finite set consisting of r^n words, so there is a one-to-one function f_1 mapping the set E_1 onto the initial segment $\{1, 2, \dots, r\}$, a one-to-one function f_2 mapping the set E_2 onto the next segment $\{1 + r, 2 + r, \dots, r + r^2\}$ of length r^2, and so forth. For each $n \geq 2$, the function f_n maps the set E_n onto the segment

$$\{1 + r + r^2 + \cdots + r^{n-1},\ 2 + r + r^2 + \cdots + r^{n-1}, \dots, r + r^2 + \cdots + r^{n-1} + r^n\},$$

of length r^n. But then the union of these functions $\bigcup_{n \in \mathbb{N}} f_n$ provides a one-to-one function from the set $\bigcup_{n \in \mathbb{N}} E_n$ onto $\mathbb{N}$; this shows that the set of all words that may be formed from a finite alphabet is countably infinite. It is elementary that a subset of a countable set is countable (see, e.g., [**2**, p. 19]), so the second assertion is immediate.

The quintessential feature of a set C that is known to be countable is that its members may be identified with words that are formable from the alphabet $\{c, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Specifically, the membership of C may be listed in a roster, which has the form $C = \{c1, c2, c3, \dots\}$ if C is infinite. The principle says that the set of *all* possible words that may be formed from this alphabet is countably infinite. Thus, elements of C constitute a distinguished subset of a larger countably infinite set that includes such elements as $3cc14$, for example. Moreover, the principle serves as a sort of converse to this quintessential feature, and so provides a *characterization* of countable sets: A set C is countable if and only if its elements may be identified with words that are formable using letters from a finite alphabet.

Examples We will use the principle to show that the following sets are countable: (i) the set $\mathbb{Q}$ of rational numbers; (ii) the set $\mathbb{Q}[x]$ of polynomials with rational coefficients; (iii) the set S of surds.

By the principle of countability, the set of all words that may be formed from the alphabet $A_1 := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /, -\}$ is countable. The set $\mathbb{Q}$ consists of objects of the form a/b, where a and b are integers and $b \neq 0$; it may be identified in an obvious way with a subset of all the words that may be so formed. Thus $\mathbb{Q}$ is countably infinite.

The members of the infinite set $\mathbb{Q}[x]$ can be thought of as words formed from the alphabet $A_2 := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, x, +, -, /\}$. For example, the polynomial $x^2 + \frac{2}{3}x - 5$ may be represented by the word $x2 + 2/3x - 5$. Hence $\mathbb{Q}[x]$ is countably infinite.

Surds are numbers that can be built using rational numbers and the basic operations of addition, subtraction, multiplication, division, and extraction of roots (see, e.g., [**3**, p. 38]). Thus any surd may be viewed as a word that may be formed using letters from the alphabet $A_3 := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -, /, (,), [,], \surd\}$. For example, the surd $\sqrt{1 + \sqrt[12]{\frac{2}{3}}}$ may be identified with the word $\surd(1 + [12]\surd(2/3))$.

The countability principle is also of theoretical interest; we apply it to prove the following theorem:

THEOREM.

(1) *A finite Cartesian product of countable sets is countable.*
(2) *A countable union of countable sets is countable.*

Proof. For (1), let S_j, $1 \leq j \leq n$, $n \in \mathbb{N}$, be a collection of countable sets. Thus, for each $1 \leq j \leq n$, there exists a one-to-one function f_j from either an initial segment of $\mathbb{N}$, or from $\mathbb{N}$ itself, onto S_j. An element of the Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ is thus an n-tuple $(f_1(m_1), f_2(m_2), \ldots, f_n(m_n))$, where $m_j \in \mathbb{N}$ and $1 \leq j \leq n$. Every such element is identifiable as a word that may be formed using letters of the alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, f, (,), \ ,\}$. (Note that this alphabet consists of 14 letters, one of which is a comma.) Thus, the principle of countability ensures that the set $S_1 \times S_2 \times \cdots \times S_n$ is countable.

For (2), we have either a finite collection S_j, $1 \leq j \leq n$, or a countably infinite collection S_j, $j \in \mathbb{N}$, of countable sets. In either case, for each index j there exists a one-to-one function f_j from either an initial segment of $\mathbb{N}$, or from $\mathbb{N}$ itself, onto S_j. Now, any element of $\bigcup_j S_j$ may be expressed as $f_k(m)$, for $k \in \mathbb{N}$ (of least index) and some $m \in \mathbb{N}$. Every such element is thus identifiable as a word that may be formed from the alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, f, (,)\}$, so the principle of countability ensures that the union $\bigcup_j S_j$ is countable.

Extensions The countability principle fails if one permits words of countably infinite length. In this case, the set of all words that may be formed from any finite alphabet with more than one letter is uncountable. For example, the set of all words formable from the finite alphabet $\{0, 1\}$ contains the uncountable set of all infinite sequences of 0's and 1's.

On the other hand, if one permits a countably infinite alphabet, but requires that words have finite length, then, as in the principle, the set of all formable words is countable. To see this, let A denote the countably infinite alphabet, and observe that the set E_n of words of length n that may be formed from A may be identified with the n-fold Cartesian product $A^n = A \times A \times \cdots \times A$. Part (1) of the theorem ensures that A^n is countable for each $n \in \mathbb{N}$. But then $\bigcup_{n \in \mathbb{N}} A^n$ is the set of all words formable using letters of A, so part (2) of the theorem ensures its countability.

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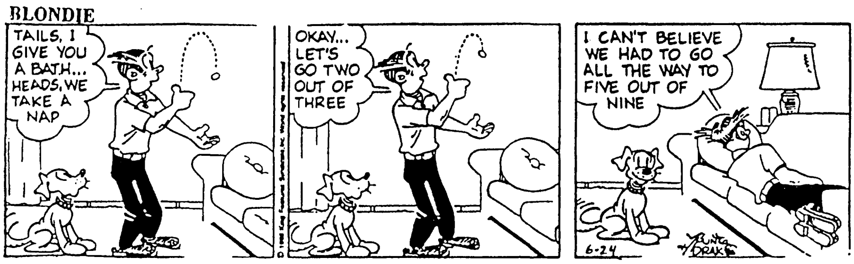
Math Bite: The Dagwood Random Nap



By tossing the coin only enough times for a majority of the outcomes to show heads, Dagwood has encountered a first-passage problem for random walk.

To formulate the random walk model corresponding to "steps" up and down with equal probabilities, assume $X_1, X_2, \ldots$ are obtained from independent and identically distributed Bernoulli trials, each with success probability $1/2$, so that $\text{Prob}(X_i = +1) = \text{Prob}(X_i = -1) = 1/2$. The corresponding (symmetric) simple random walk is then $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \ldots$.

Because S_n records the difference between the number of heads and the number of tails observed in the first n tosses, Dagwood's first passage problem then involves the random time $T_{\{1\}} = \min\{n \geq 1 | S_n = 1\}$. Observing that the event $\{S_n = 1\}$ can occur only for an odd number of tosses, we wish to calculate $\text{Prob}(T_{\{1\}} = 2n - 1)$ and $\text{Prob}(T_{\{1\}} > 2n - 1)$ for $n = 1, 2, \ldots$.

Explicit first-passage solutions are given for $n \geq 1$ by

$$\text{Prob}\left(T_{\{1\}} = 2n - 1\right) = \frac{1}{2n-1}\binom{2n-1}{n}\frac{1}{2^{2n-1}};$$

$$\text{Prob}\left(T_{\{1\}} > 2n - 1\right) = \binom{2n}{n}\frac{1}{2^{2n}}.$$

These formulas show that $\text{Prob}(T_{\{1\}} = 9) = 14 \times 1/2^9 \approx 0.027$ and $\text{Prob}(T_{\{1\}} > 9) = 252 \times 1/2^{10} \approx 0.246$. The (perhaps more relevant) second probability figure shows that nearly one in four Dagwood naps would require more than 9 coin tosses.

The first-passage probability solutions can be derived either from the analytic treatment in [**1**, p. 76–77] or the insightful geometric argument in [**2**]. Finally, it can be shown that a Dagwood random nap is indeed certain, but that the expected number of tosses is infinite [**1**, p. 272].

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A Combinatorial Approach to Sums of Integer Powers

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Introduction The fact that $S_p(n) := \sum_{k=1}^{n} k^p$, the sum of the pth powers of the first n positive integers, can be written as a polynomial in n for any positive integer p is certainly not a state secret. This topic has been treated extensively over the years from various perspectives, with **[1]**, **[3]**, **[5]**, **[6]**, and **[7]** examples of recent contributions. Evidence provided by small values of p, such as $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, makes this fact plausible even to the beginner and might also spark the conjecture that the resulting polynomial should have degree $p+1$. Proofs that validate this or produce a polynomial expression for $\sum_{k=1}^{n} k^p$ often tend to use mathematical induction or recursion. Valuable though they are, such inductive or recursive techniques can sometimes disguise underlying motivation and perhaps leave a reader with a vague feeling of not having grasped the heart of the matter.

In this note we offer a combinatorial interpretation of $S_p(n)$ that can serve to motivate why this sum can be expressed as a polynomial of degree $p+1$ in the variable n. Our approach allows us to make some general statements about coefficients of this polynomial and also produces a technique for direct calculation of the polynomial. We show that calculating this polynomial can be accomplished by solving a $(p-4)\times(p-4)$ lower triangular system of linear equations. Our exposition requires only the ability to use binomial coefficients in basic counting arguments.

A combinatorial interpretation A common technique in combinatorics is to count something in two different ways and equate the answers, thus deriving an identity. Here, we search for a set of objects with cardinality $S_p(n)$. We choose the set of those vectors $(x_1, x_2, \dots, x_{p+1})$, with positive integer components, satisfying

$$1 \le x_i \le n+1, \; i = 1, \dots, p+1 \quad \text{and} \quad x_1 > x_i, \; i = 2, \dots, p+1.$$

For any fixed allowable value of x_1, say $x_1 = j$, there are clearly $(j-1)^p$ vectors satisfying the above condition, with $x_1 = j$. Since x_1 can be any integer between 2 and $n+1$, there are a total of $\sum_{j=2}^{n+1}(j-1)^p = \sum_{k=1}^{n} k^p$ such vectors.

An alternative way of counting these vectors is to focus on the number of distinct positive integers among the components, $x_1, x_2, \dots, x_{p+1}$. How many vectors have all $p+1$ components distinct? There are $\binom{n+1}{p+1}$ ways of choosing the integers that will occupy the components. Since the only requirement is that the largest of these integers occupy the first position, all $p!$ permutations of the remaining integers can be used to fill in the other components. There are thus $\binom{n+1}{p+1}p!$ such vectors.

The number of vectors with p distinct integers among their components is $\binom{n+1}{p}\binom{p-1}{1}\frac{p!}{2}$, since, of the $p-1$ integers available for positions 2 through $p+1$, one of these must be chosen to be repeated twice, and the resulting collection of integers with repetition can be permuted in $\frac{p!}{2}$ ways.

When counting vectors having $p-1$ distinct integers among their components, we must distinguish between the two ways this can happen: either one integer is repeated three times in the last p positions, or else two different integers each occur twice. Keeping this in mind, we count

$$\binom{n+1}{p-1}\left[\binom{p-2}{1}\frac{p!}{3!}+\binom{p-2}{2}\frac{p!}{2!2!}\right]$$

vectors of this type.

While it is difficult in general to write down a closed form expression for the number of vectors having t distinct integers among their components, our argument does show that this number is of the form $\binom{n+1}{t}c_t$, where c_t is a non-negative integer.

Combining our two points of view, we see that

$$S_p(n)=\sum_{k=1}^{n}k^p=\binom{n+1}{p+1}p!+\sum_{t=2}^{p}c_t\binom{n+1}{t},$$

for some integers c_t. (There is no term corresponding to $t=1$, since the vectors being counted must contain at least two distinct integers.) Note that we have already determined the values of c_p and c_{p-1}. Our work so far allows us to make the following observations:

1. Viewed as an expression in n, $\binom{n+1}{t}$ is a polynomial of degree t. Also, the coefficient of n^{p+1} in the expansion of $\binom{n+1}{p+1}p!$ is $\frac{1}{p+1}$. It follows that $S_p(n)$ is a polynomial in n of degree $p+1$ with rational coefficients and leading coefficient $\frac{1}{p+1}$.
2. Since, for $2\le t\le p+1$, $\binom{n+1}{t}$ contains the term $n(n+1)$, we see that $S_p(n)$ always has $n(n+1)$ as a factor. Our first two observations combine then to force the familiar result $S_1(n)=n(n+1)/2$.
3. The coefficient of n^p in $S_p(n)$ is always $\frac{1}{2}$. This can be readily seen by computing this coefficient in $\binom{n+1}{p+1}p!+\binom{n+1}{p}\binom{p-1}{1}\frac{p!}{2}$.

Thus, for example, we must have $S_2(n)=\frac{1}{3}n^3+\frac{1}{2}n^2+cn$, for some integer c. Since $S_2(1)=1$, we solve for c and find that $S_2(n)=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$.

Direct calculation Our approach will permit direct calculation of the polynomial

$$S_p(n)=\binom{n+1}{p+1}p!+\binom{n+1}{p}\binom{p-1}{1}\frac{p!}{2}+\sum_{t=2}^{p-1}c_t\binom{n+1}{t},$$

provided we can effectively calculate the various integers c_t. Recall that c_t is the number of vectors having p components that are possible if we insist that the entries come from $t-1$ previously chosen integers. Thus, it is always the case that $c_2=1$.

Also, we have already shown that

$$c_{p-1}=\binom{n+1}{p-1}\left[\binom{p-2}{1}\frac{p!}{3!}+\binom{p-2}{2}\frac{p!}{2!2!}\right].$$

We will use the case $p=6$ to illustrate how to find the remaining c_t's.

Now,

$$S_6(n) = \sum_{k=1}^{n} k^6 = \binom{n+1}{7}6! + \binom{n+1}{6}\binom{5}{1}\frac{6!}{2!} + \binom{n+1}{5}\left[\binom{4}{1}\frac{6!}{3!} + \binom{4}{2}\frac{6!}{2!2!}\right]$$
$$+ c_4\binom{n+1}{4} + c_3\binom{n+1}{3} + \binom{n+1}{2}.$$

An interesting way to find c_3 would be to replace n by 2 in the above equation and take advantage of the fact that the binomial coefficient $\binom{a}{b}$ is zero whenever b is greater than a, while $\binom{a}{a} = 1$. The result is $c_3 = S_6(2) - \binom{3}{2} = 1^6 + 2^6 - 3 = 62$. Similarly, letting $n = 3$, we get $c_4 = S_6(3) - \binom{4}{3}c_3 - \binom{4}{2} = 540$. If desired, we can expand binomial coefficients and collect terms to arrive at

$$S_6(n) = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n.$$

In general, replacing n successively by $2, 3, \ldots,$ and $p-3$ in our expression for $S_p(n)$ will produce a system of linear equations in the unknowns $c_3, c_4, \ldots, c_{p-2}$. Since c_t occurs multiplied by $\binom{n+1}{t}$, this $(p-4) \times (p-4)$ system is lower triangular, with ones on the diagonal. Solving it will yield a polynomial expression for $S_p(n)$.

Concluding remarks In the article **[6]**, Paul also uses a counting argument involving lattice points, though the set of points counted and the conclusions drawn differ from those in this note. The topics discussed here are linked to many interesting developments in the history of computation. James Bernoulli (1654–1705) worked on the problem of finding formulas for sums of pth powers of integers, in the process constructing what we now term *Bernoulli numbers* **[8]**. Also, the coefficients c_t introduced above are close relatives of another famous sequence of numbers, the *Stirling numbers*, $S(n, m)$, *of the second kind*, which count the number of ways to partition a set of n elements into m nonempty subsets (**[2]**, **[4]**).

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Rounding Errors to Knock Your Stocks Off

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Introduction As the total value of all daily stock transactions on the Vancouver Stock Exchange kept rising in 1982 and 1983, the exchange's index kept falling. How could this be happening?

The cause of the divergence between the mathematical and computed averages consisted in an erroneous algorithm to round the last digits during the computation of the index. Public documents, e.g., Quinn's article **[2]** include the following pieces of information. The index is the arithmetic average of the selling prices of the nearly 1400 stocks listed on the Vancouver Stock Exchange. The computation of the index started in January of 1982, with the index then pegged to 1000. In about November of 1983, exchange officials estimated that the index should have been at least at 900 and perhaps above 1000, but the computed value of the index was down near 520. The index was computed every time the price of a stock changed, which occurred about 2800 times per day. The computer carried a total of eight decimal digits during the computation, but it truncated the last two digits to display and record the index with only three decimal digits past the decimal point. Thus, if it computed the value 540.32567, then it would record 540.325 for the index.

The magnitude of the discrepancy—about 520 instead of 1000 or so—indicated that the cause involved more than only an erroneous rounding of an otherwise correct computation. Indeed, with a computer carrying eight decimal digits, the relative rounding error caused by each addition cannot exceed one half of one unit in the last digit, which is $(1/2) \times 10^{1-8}$. In the worst case, if all 1399 additions of all 1400 stock prices suffered from the maximum relative error, errors would compound to at most

$$\left[1 + \tfrac{1}{2} \times 10^{7}\right]^{1399} - 1 = 0.000\,069\,952\ldots < 10^{-4},$$

so that the computed value of the index would contain an error equal to at most one unit in the fourth significant digit. For example, if the index had the value 1000, then the computed value would still lie between 999.9 and 1000.1.

What operations inside the computer could produce 520? The *increasing* magnitude of the discrepancy—with stock prices soaring and the index sinking—suggests errors that compound and perpetuate themselves from one computation of the index to the next computation. The mathematical definition of the average $\bar{X}$ of N stock prices $X_1, \ldots, X_N$ uses the formula

$$\bar{X} = \{[(X_1 + X_2) + X_3] + \cdots + X_N\}/N,$$

where parentheses indicate the sequence in which the computer performs the operations. An electronic computer could compute such an average of $N = 1400$ numbers 2800 times daily. However, if in an instant stock prices change more quickly than the computer can compute the average, an alternate *updating* algorithm allows for the computation of the change in the average without computing the average all over again. Specifically, if the Nth stock price changes from X_N to X'_N (the ordering of the stocks does not matter, because addition commutes), then the average changes from $\overline{X}$ to $\overline{X}'$ as follows:

$$\overline{X}' = \frac{1}{N}\left(\sum_{J=1}^{N-1} X_J + X_N'\right) = \overline{X} + \frac{1}{N}(X_N' - X_N).$$

Thus, to compute the new average $\overline{X}'$, it suffices to add $(X_N' - X_N)/N$ to the old average $\overline{X}$. This alternate algorithm merely updates the old average with the change in the stock price to produce the new average. Instead of the $N - 1$ additions involved in the definition of the average, the updating algorithm involves only one subtraction $(X_N' - X_N)$ and one addition. (The division by N occurs with both algorithms.)

How could a simpler algorithm lead to larger errors? Updates are mathematically exact but very sensitive to rounding errors. Recall that the computer truncated the last two digits from the computed average to display only three digits past the decimal point. In effect, the computer was replacing the average $\overline{X}$ by a truncated value $\hat{X}$, which caused an error in the range

$$0.000\,00 \leq \overline{X} - \hat{X} \leq 0.000\,99.$$

If all possible errors occur equally frequently, then the average error amounts to

$$\frac{0.000\,00 + 0.000\,01 + \cdots + 0.000\,98 + 0.000\,99}{100} = 0.000\,455.$$

Chopping 0.000 455 off the index 2800 times in a day accumulates a daily drop in the index of over one point: $2800 \times 0.000\,455 = 1.274$. Repeated over about 480 business days from January of 1982 through the first week of November of 1983, rounding by truncation of the last two digits results in an accumulated rounding error of $480 \times 1.274 = 611.52$. This explains the discrepancy between an estimated value in the range 900 to 1000 or more and a computed value around 520, from January of 1982 to November of 1983.

Could such discrepancies have been avoided? Yes, because averaging 1400 *non-negative* numbers from the definition of the average (*not* the updating algorithm) through floating-point arithmetic with 8 digits yields a computed average $\tilde{X}$ accurate to nearly 4 digits. Better yet, pairwise additions would produce nearly 6 accurate digits [**1**, p. 91].

In this instance many investors and officials had been watching stock prices closely and hence had an inkling that the results were skewed. Beside stock prices, computers also compute many other indicators, for instance, consumer price indices, poverty levels, and hence levels of government assistance and taxation. Fortunately, with today's emphasis on technology in the classroom unlike anything seen in 1982, our students are aware of the potential consequences of rounding errors in the last digits, aren't they?

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2. K. Quinn, Ever had problems rounding off figures? This stock exchange has, *The Wall Street Journal*, Eastern Ed., Vol. CCII, No. 91 (Tuesday 8 November 1983), Section 2, p. 37.

If $d \le r$, the rational number $[a_0; a_1, a_2, a_3, \dots, a_d]$ is called the d^{th} *convergent* to the continued fraction $[a_0; a_1, a_2, a_3, \dots, a_r]$. For example, $[1; \underbrace{1,1,1,1,\dots,1}_{r}]$ has convergents

$$[1] = 1,\ [1;1] = \frac{2}{1},\ [1;1,1] = \frac{3}{2},\ [1;1,1,1] = \frac{5}{3},\ [1;1,1,1,1] = \frac{8}{5}, \dots .$$

As one might guess from these results, the d^{th} convergent satisfies

$$\Big[1; \underbrace{1,1,1,1,\dots,1}_{d}\Big] = \frac{F_{d+2}}{F_{d+1}},$$

where F_d is the d^{th} Fibonacci number. An infinite continued fraction $[a_0; a_1, a_2, a_3, \dots]$ is defined as the limit of its convergents. For example,

$$[1;1,1,\dots] = \lim_{d\to\infty} \frac{F_{d+2}}{F_{d+1}} = \frac{1+\sqrt{5}}{2}.$$

The continued fraction expansion of any positive number z can be obtained by setting $a_0 = \lfloor z \rfloor$ (where $\lfloor z \rfloor$ is the greatest integer not exceeding z) and iterating the function $f(t) = \frac{1}{t - \lfloor t \rfloor}$. If z is rational, the iteration eventually produces an integer, and the expansion is complete. We will use the following properties of continued fractions in what follows. We assume throughout that x is a given irrational number and $x = [a_0; a_1, a_2, a_3, \cdots]$ is its continued fraction.

Fact 1 The convergents $\frac{p_r}{q_r} = [a_0; a_1, a_2, a_3, \dots, a_r]$ to the continued fraction $[a_0; a_1, a_2, a_3, \dots]$ can be computed for $r \ge 0$ by the recursive formula

$$(p_{r+1}, q_{r+1}) = a_{r+1}(p_r, q_r) + (p_{r-1}, q_{r-1})$$

with initial conditions $(p_{-1}, q_{-1}) = (1, 0)$ and $(p_0, q_0) = (a_0, 1)$.

Fact 2 $e = [2; 1,2,1,1,4,1,1,6,1,1,8,1,1,10,\dots]$. Using Fact 1 to compute the first few convergents to e (see table below) we see by induction that p_r is even and q_r is odd if and only if $r \equiv 0$ or $r \equiv 2 \pmod 6$.

i	0	1	2	3	4	5	6	7
$\frac{p_i}{q_i}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{8}{3}$	$\frac{11}{4}$	$\frac{19}{7}$	$\frac{87}{32}$	$\frac{106}{39}$	$\frac{193}{71}$

Fact 3 Let a'_{r+1} denote $[a_{r+1}; a_{r+2}, a_{r+3}, \dots]$. Then

$$x - \frac{p_r}{q_r} = \frac{(-1)^r}{q_r^2} \frac{1}{a'_{r+1} + \frac{q_{r-1}}{q_r}}.$$

It follows that the convergents p_n/q_n lie alternately above and below x.

Fact 4 If $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$, for relatively prime integers p and q, then p/q is a convergent to the continued fraction of x.

We now use these facts to study rational approximations of e.

LEMMA 1. *Let p_r/q_r be the rth convergent to the continued fraction for e. If $r \equiv 0$ or $r \equiv 2 \pmod 6$, then*

$$q_r |q_r e - p_r| \ge \frac{6}{13}.$$

Proof. By Fact 3, it suffices to show $a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{13}{6}$. If $r = 6j$, then

$$a'_{r+1} = [1; 4j+2, 1, 1, \dots] < 1 + \frac{1}{4j+2} \le \frac{7}{6},$$

so

$$a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{7}{6} + 1 = \frac{13}{6}.$$

If $r = 6j + 2$, then $a'_{r+1} = [1; 1, 4j+4, 1, \dots] < 2$. By Facts 1 and 2,

$$\frac{q_{r-1}}{q_r} = \frac{q_{r-1}}{(4j+2)q_{r-1} + q_{r-2}} < \frac{1}{4j+2} \le \frac{1}{6},$$

so

$$a'_{r+1} + \frac{q_{r-1}}{q_r} < \frac{13}{6}. \qquad \blacksquare$$

LEMMA 2. *For all* $t > 0$, $\dfrac{(t+1)^{t+1}}{t^t} \le e\left(t + \dfrac{1}{2}\right)$.

Proof. If $h(x)$ is a concave function, integrable on $[a, b]$, then Jensen's inequality for integrals gives $h\left(\frac{a+b}{2}\right) \ge \frac{1}{b-a}\int_a^b h(x)\,dx$. Take $h(x) = \ln x$ and $[a, b] = [t, t+1]$ to get

$$\ln\left(t + \frac{1}{2}\right) \ge (t+1)\ln(t+1) - t\ln t - 1,$$

which is equivalent to the lemma. $\blacksquare$

Proof of the Theorem Recall that $f(n) = \left(\frac{k}{n}\right)^n$. By examining the first derivative of $\ln f(n)$, we see that the maximal product occurs either at $n = \left\lfloor \frac{k}{e} \right\rfloor$ or at $\left\lfloor \frac{k}{e} \right\rfloor + 1$. Let $t = \left\lfloor \frac{k}{e} \right\rfloor$. Direct calculation verifies the theorem for $k \le 71$, so we may assume that $k > 71$, or, equivalently, that $t \ge 27$. If $\frac{k}{e} > t + \frac{1}{2}$ then $\left|\frac{k}{e} - n\right|$ is minimized when $n = t + 1$. By Lemma 2,

$$k > e\left(t + \tfrac{1}{2}\right) > \frac{(t+1)^{t+1}}{t^t},$$

which implies that $f(t+1) > f(t)$, as desired.

Now suppose that $\frac{k}{e} < t + \frac{1}{2}$. Let $s = t + \frac{1}{2} - \frac{k}{e} > 0$. By Fact 2 and Lemma 3,

$$\frac{6}{13} \le q_n |q_n e - p_n| = (2t+1)|(2t+1)e - 2k| = 2e(2t+1)s.$$

Since $t \ge 27$, we have $s \ge \frac{3}{13e(2t+1)} > \frac{1}{24t}$, and so $k < \left(t + \frac{1}{2} - \frac{1}{24t}\right)e$. Dividing by $t + 1$,

$$\frac{k}{t+1} < e\left(\frac{t + \frac{1}{2} - \frac{1}{24t}}{t+1}\right) = e\left(1 - \frac{12t+1}{24t(t+1)}\right).$$

Taking logs and writing $\ln(1+z)$ as an alternating series gives

$$\ln\frac{k}{t+1} < 1 + \ln\left(1 - \frac{12t+1}{24t(t+1)}\right) = 1 - \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{12t+1}{24t(t+1)}\right)^k.$$

Therefore,

$$t\ln\left(1+\frac{1}{t}\right)-\ln\frac{k}{t+1}>t\sum_{k=1}^{4}\frac{(-1)^{k-1}}{kt^k}-1+\sum_{k=1}^{3}\frac{1}{k}\left(\frac{12t+1}{24t(t+1)}\right)^k$$
$$=\frac{864t^3-7308t^2-17208t-10367}{41472t^3(t+1)^3}.$$

The numerator has only one real root, at $t\approx 10.47$, and is therefore positive since $t\geq 27$. Thus $f(t)>f(t+1)$. ■

Counterexamples to the conjecture A computer search found the first twelve counterexamples to the conjecture: 53, 246, 439, 632, 12973, 62144, 111315, 160486, 209657, 258828, 7332553, 205052656.

In fact, there are infinitely many counterexamples. Suppose k is a counterexample, and let n be the integer that minimizes $|\frac{k}{n}-e|$. Then e must lie between $\frac{k}{n}$ and $\frac{k}{n+1}$ or between $\frac{k}{n-1}$ and $\frac{k}{n}$. If $\frac{k}{n+1}<e<\frac{k}{n}$, then $\frac{2e}{k}>\frac{2n+1}{n(n+1)}$. Since n does not minimize $|\frac{k}{e}-n|$, $\frac{k}{e}$ is closer to $n+1$ than to n, whence $2\frac{k}{e}>2n+1$. Multiplying these last two inequalities yields $4>\frac{(2n+1)^2}{n^2+n}$, a contradiction. Hence $\frac{k}{n}<e<\frac{k}{n-1}$. In this case, $\frac{2e}{k}>\frac{2n-1}{n(n-1)}$ and $\frac{2k}{e}<2n-1$, so k is a counterexample if and only if

$$0<e-\frac{2k}{2n-1}<\frac{e}{(2n-1)^2}. \tag{1}$$

By Fact 3, the right-hand inequality in (1) is satisfied by each convergent p_r/q_r to e. By Fact 2, there are integers k and n with $\frac{p_r}{q_r}=\frac{2k}{2n-1}$ if $r\equiv 0$ or $r\equiv 2 \pmod 6$. By Fact 3, the successive convergents must be alternately above and below e. Since $p_0/q_0=2/1<e$, we have $p_r/q_r<e$ for all even r. Hence (1) holds for $2k=p_r$ and $2n-1=q_r$, where $r\equiv 0$ or $r\equiv 2 \pmod 6$. This method produces infinitely many counterexamples of the form $k=p_{6j}/2$ or $k=p_{6j+2}/2$, for $j=1,2,3,\ldots$. These account for six of the first twelve counterexamples, mentioned earlier:

j	1	2	3
$p_{6j}/2$	53	12973	7332553
$p_{6j+2}/2$	632	258828	205052656

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Triangles with Integer Sides, Revisited

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Introduction Our attention has once more been drawn (see [**2**] and its references) to the problem of determining the number T_n of triangles with integer sides and perimeter n. The solution of this problem can be written neatly as

$$T_n = \begin{cases} \left\langle \dfrac{(n+3)^2}{48} \right\rangle & \text{for } n \text{ odd} \\ \left\langle \dfrac{n^2}{48} \right\rangle & \text{for } n \text{ even,} \end{cases}$$

where $\langle x \rangle$ is the integer closest to x. Our proof comes in two stages. First we show by a direct combinatorial argument that

$$T_n = \begin{cases} p_3\left(\dfrac{n-3}{2}\right) & \text{for } n \text{ odd} \\ p_3\left(\dfrac{n-6}{2}\right) & \text{for } n \text{ even,} \end{cases}$$

where $p_3(n)$ is the number of partitions of n into at most three parts. Then we show using a novel partial fractions technique that

$$p_3(n) = \left\langle \frac{(n+3)^2}{12} \right\rangle.$$

The proofs For a triangle with integer sides $a \le b \le c$ and odd perimeter n,

$$a+b-c, \quad b+c-a, \quad c+a-b$$

are odd and positive,

$$a+b-c-1, \quad b+c-a-1, \quad c+a-b-1$$

are even and nonnegative, and if

$$p = \frac{1}{2}(b+c-a-1), \quad q = \frac{1}{2}(c+a-b-1), \quad \text{and} \quad r = \frac{1}{2}(a+b-c-1),$$

then $p \ge q \ge r$ are nonnegative integers, and $p+q+r = \dfrac{n-3}{2}$.

Conversely, if $n \ge 3$ is odd and $p \ge q \ge r$ are nonnegative integers with $p+q+r = \dfrac{n-3}{2}$ and if

$$a = q+r+1, \quad b = p+r+1, \quad \text{and} \quad c = p+q+1,$$

then $a \le b \le c$ are the sides of a triangle with perimeter n.

Similarly, given a triangle with integer sides $a \le b \le c$ and even perimeter n,

$$a+b-c, \quad b+c-a, \quad \text{and} \quad c+a-b$$

are even and positive,

$$a+b-c-2, \quad b+c-a-2, \quad \text{and} \quad c+a-b-2$$

are even and nonnegative, and if

$$p=\frac{1}{2}(b+c-a-2),\quad q=\frac{1}{2}(c+a-b-2),\quad \text{and}\quad r=\frac{1}{2}(a+b-c-2),$$

then $p \geq q \geq r$ are nonnegative integers, and $p+q+r=\dfrac{n-6}{2}$.

Conversely, if $n \geq 6$ is even and $p \geq q \geq r$ are nonnegative integers with $p+q+r = \dfrac{n-6}{2}$ and if

$$a=q+r+2,\quad b=p+r+2,\quad \text{and}\quad c=p+q+2$$

then $a \leq b \leq c$ are the sides of a triangle with perimeter n. ■

To show that

$$p_3(n)=\left\langle \frac{(n+3)^2}{12}\right\rangle,$$

we start with the generating function

$$\sum_{n\geq 0} p_3(n)q^n=\frac{1}{(1-q)(1-q^2)(1-q^3)}.$$

To see that this is indeed the generating function for partitions into at most three parts, we note that partitions into at most three parts are equinumerous with partitions into parts no greater than three (see, e.g., [**1**, Theorem 1.4]) and the generating function for partitions into parts no greater than three is easily seen to be

$$\begin{aligned}&(1+q^3+q^{3+3}+\cdots)(1+q^2+q^{2+2}+\cdots)(1+q^1+q^{1+1}+\cdots)\\ &=\frac{1}{(1-q)(1-q^2)(1-q^3)}.\end{aligned}$$

In order to extract an explicit formula for $p_3(n)$ from the generating function, it is usual to use partial fractions. I have found a method (other than by using a computer package, such as *Maple*) by which we can avoid the horrors of finding the complete partial fractions expansion of the generating function. I will demonstrate the method below, and follow it with an explanation of the various steps.

First, observe that the denominator of the generating function can be factored as

$$(1-q)^3(1+q)(1-\omega q)(1-\overline{\omega} q),$$

where ω is a cube root of unity. Thus the partial fractions expansion of the generating function takes the form

$$\sum_{n\geq 0} p_3(n)q^n=\frac{A}{(1-q)^3}+\frac{B}{(1-q)^2}+\frac{C}{1-q}+\frac{D}{1+q}+\frac{E}{1-\omega q}+\frac{F}{1-\overline{\omega} q}.$$

It follows that

$$p_3(n)=A\binom{n+2}{2}+B\binom{n+1}{1}+C+D(-1)^n+E\omega^n+F\overline{\omega}^n.$$

Observe that the expression $C+D(-1)^n+E\omega^n+F\overline{\omega}^n$ is periodic with period 6, and so takes values c_i, $i=0,\ldots,5$, according to the residue of n modulo 6. It follows that the generating function can be written

$$\sum_{n\geq 0} p_3(n)q^n=\frac{A}{(1-q)^3}+\frac{B}{(1-q)^2}+\frac{c_0+c_1q+c_2q^2+c_3q^3+c_4q^4+c_5q^5}{1-q^6}.$$

With these observations in mind, we have

$$\begin{aligned}
\sum_{n\ge 0} p_3(n)q^n &= \frac{1}{(1-q)(1-q^2)(1-q^3)} \\
&= \frac{1}{(1-q)^3}\cdot\frac{1}{(1+q)(1+q+q^2)} \\
&= \frac{1}{(1-q)^3}\cdot\left(\frac{1}{6}+\frac{(1-q)(5+3q+q^2)}{6(1+q)(1+q+q^2)}\right) \\
&= \frac{1}{6}\frac{1}{(1-q)^3}+\frac{1}{(1-q)^2}\cdot\frac{5+3q+q^2}{6(1+q)(1+q+q^2)} \\
&= \frac{1}{6}\frac{1}{(1-q)^3}+\frac{1}{(1-q)^2}\cdot\left(\frac{1}{4}+\frac{(1-q)(7+7q+3q^2)}{12(1+q)(1+q+q^2)}\right) \\
&= \frac{1}{6}\frac{1}{(1-q)^3}+\frac{1}{4}\frac{1}{(1-q)^2}+\frac{7+7q+3q^2}{12(1-q)(1+q)(1+q+q^2)} \\
&= \frac{1}{6}\frac{1}{(1-q)^3}+\frac{1}{4}\frac{1}{(1-q)^2}+\frac{(7+7q+3q^2)(1-q+q^2)}{12(1-q^6)} \\
&= \frac{1}{6}\frac{1}{(1-q)^3}+\frac{1}{4}\frac{1}{(1-q)^2}+\frac{7+3q^2+4q^3+3q^4}{12(1-q^6)} \\
&= \frac{1}{6}\sum_{n\ge 0}\binom{n+2}{2}q^n+\frac{1}{4}\sum_{n\ge 0}(n+1)q^n \\
&\quad +\frac{1}{12}(7+3q^2+4q^3+3q^4)\sum_{n\ge 0}q^{6n} \\
&= \sum_{n\ge 0}\frac{1}{12}(n^2+6n+5)q^n+\frac{1}{12}(7+3q^2+4q^3+3q^4)\sum_{n\ge 0}q^{6n} \\
&= \sum_{n\ge 0}\frac{(n+3)^2}{12}q^n+\frac{1}{12}(3-4q-q^2-q^4-4q^5)\sum_{n\ge 0}q^{6n}.
\end{aligned}$$

The result follows.

The explanation of the various steps above is as follows. Since we know the major contribution to $p_3(n)$ comes from the term $(1-q)^3$ in the denominator, we attempt to separate this term from the generating function. We write

$$\sum_{n\ge 0} p_3(n)q^n = \frac{1}{(1-q)^3}f(q), \quad \text{where} \quad f(q)=\frac{1}{(1+q)(1+q+q^2)}.$$

Then we replace $f(q)$ by its Taylor series about 1, at least to the extent of writing

$$f(q)=f(1)+g(q)=\frac{1}{6}+g(q).$$

An easy calculation gives

$$g(q) = f(q) - f(1) = \frac{1}{(1+q)(1+q+q^2)} - \frac{1}{6}$$
$$= \frac{6-(1+q)(1+q+q^2)}{6(1+q)(1+q+q^2)} = \frac{(1-q)(5+3q+q^2)}{6(1+q)(1+q+q^2)}.$$

Thus we obtain the fourth line above.

Applying the same procedure to the function $\frac{5+3q+q^2}{6(1+q)(1+q+q^2)}$, we arrive at the sixth line, where the denominator is $12(1-q)(1+q)(1+q+q^2)$. Now we note that $(1-q)(1+q)(1+q+q^2)$ divides $1-q^6$, since

$$1-q^6 = (1-q^3)(1+q^3) = (1-q)(1+q)(1+q+q^2)(1-q+q^2).$$

So we multiply top and bottom by $1-q+q^2$ to obtain the eighth line above. The rest is straightforward.

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Math Bite: Enumerating Certain Sparse Matrices

THEOREM. *The number of $n \times n$ matrices with (i) nonnegative integer entries; (ii) at most two nonzero entries in each line (i.e., row or column); (iii) all line-sums* 3; *is* $n!^2$.

Proof.

$$\begin{bmatrix} 0 & 0 & 3 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(typical matrix) (erase all the 2's, change 3's to 1's) (erase all the 1's, change 3's to 2's)

Remark The $n \times n$ matrices satisfying (i) and (ii) above, but with all line-sums equal to r, are enumerated using generating functions in the recently-published *Enumerative Combinatorics, Vol.* 2, by Richard Stanley (Problem 5.62). The generating function yields the surprisingly simple formula $n!^2$ in the special case $r = 3$. The picture above gives a combinatorial explanation; it shows that such a matrix can be uniquely represented as $P + 2Q$, where P and Q range over all arbitrary $n \times n$ permutation matrices.

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How Many Magic Squares Are There?

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Only the crustiest of curmudgeons doesn't love magic squares. And even these sour souls know some folklore about magic squares. But how many of us know how many magic squares there are? In [**3**] Liang-shin Hahn posed the question: How many 4×4 multiplicative magic squares are there that consist of the 16 divisors of 1995. In this brief note we answer Hahn's question by establishing a natural bijection between a class of multiplicative magic squares and a class of additive magic squares (cf. [**1**]).

A multiplicative (respectively, additive) *magic square* is an $n \times n$ matrix of integers in which the product (respectively, sum) of the numbers in each row, in each column and in each diagonal is the same. An additive magic square of *order* n is an $n \times n$ additive magic square whose entries consist of the numbers $0, 1, \ldots, n^2 - 1$. It is easy to show that the sum of the numbers in each row, in each column and in each diagonal of an additive magic square of order n is equal to $(n/2)(n^2 - 1)$. Much is known about additive magic squares [**2**], while surprisingly little has been written about multiplicative magic squares. We offer the present paper as a modest remedy to this situation.

Let c be the product of n distinct prime numbers. Thus, c has 2^n factors. For example, 1995 is the product of four primes: $3, 5, 7, 19$, and has sixteen factors: $1, 3, 5, 7, 15, 19, 21, 35, 57, 95, 105, 133, 285, 399, 665, 1995$. If n is even, let $M_{c,n}$ be the set of all $2^{n/2} \times 2^{n/2}$ multiplicative magic squares each of whose entries consists of the 2^n factors of c. Let A_n be the set of all additive magic squares of order n.

The following facts are easy to establish:

FACTS. *Let M be a magic square in $M_{c,n}$. Then*

i. *The product of all 2^n factors of c is $c^{2^{n-1}}$. Thus, the product of each row, each column, and each diagonal of M is $(c^{2^{n-1}})^{2^{-n/2}} = (c^2)^{(n/2)-1}$. That is, M has precisely $2^{(n/2)-1}$ occurrences of each of the n prime factors of c in each row, in each column, and in each diagonal.*

ii. *No single prime factor of c occurs more than once in any given entry in M.*

THEOREM. *$M_{c,n}$ is in 1-1 correspondence with $A_{2^{n/2}}$.*

Proof. Let $M = [m_{i,j}]$ be a magic square in $M_{c,n}$. By the Facts, each $m_{i,j}$ can be identified with a unique string of length n consisting of 0's and 1's as follows: arrange the prime factors of c in ascending order $p_1, p_2, \ldots, p_n$; if p_k is a factor of $m_{i,j}$ place a 1 in the string's k^{th} position; if not, place a 0. For example, if $c = 1995$, and $m_{i,j} = 3 \cdot 5 \cdot 19$, then the string identified with $m_{i,j}$ is 1101. Now define $f: M_{c,n} \to A_n$; $[m_{i,j}] \mapsto [a_{i,j}]$ where $a_{i,j}$ is the base 10 number—between 0 and $n^2 - 1$—whose base 2 expression is the n-string associated with $m_{i,j}$. Clearly, $[a_{i,j}]$ is a $2^{n/2} \times 2^{n/2}$ matrix whose 2^n entries are the integers $0, 1, \ldots, 2^n - 1$. Since each prime divisor of c occurs exactly $2^{(n/2)-1}$ times in each row, in each column and in each diagonal, the sum of the strings in each row, in each column, and in each diagonal is $\sum_{i=1}^{2^{(n/2)-1}} 11 \cdots 1$; in base 10:

$$2^{(n/2)-1}(2^0 + 2^1 + \cdots + 2^{n-1}) = 2^{(n/2)-1}[(1/4)(n)(n^2-1)] = (1/2)(n)(n^2-1).$$

And thus, $[a_{i,j}]$ is an additive magic square. Clearly, f is injective. Checking that f is also surjective is left to the reader. This completes the proof.

Example. Let $c = 1995$ and consider the action of f on the following magic square M in $M_{1995,4}$:

$$f(M) = f\left(\begin{bmatrix} 21 & 1995 & 5 & 19 \\ 1 & 95 & 105 & 399 \\ 285 & 3 & 133 & 35 \\ 665 & 7 & 57 & 15 \end{bmatrix}\right)$$

$$= f\left(\begin{bmatrix} 3\cdot5 & 3\cdot5\cdot7\cdot19 & 5 & 19 \\ 1 & 5\cdot19 & 3\cdot5\cdot7 & 3\cdot7\cdot19 \\ 3\cdot5\cdot19 & 3 & 7\cdot19 & 5\cdot7 \\ 5\cdot7\cdot19 & 7 & 3\cdot19 & 3\cdot5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1010 & 1111 & 0100 & 0001 \\ 0000 & 0101 & 1110 & 1011 \\ 1101 & 1000 & 0011 & 0110 \\ 0111 & 0010 & 1001 & 1100 \end{bmatrix} = \begin{bmatrix} 10 & 15 & 4 & 1 \\ 0 & 5 & 14 & 11 \\ 13 & 8 & 3 & 6 \\ 7 & 2 & 9 & 12 \end{bmatrix}.$$

Since $|A_4| = 880$ **[2]**, the theorem answers Hahn's question and its natural generalization: $|M_{c,4}| = |A_4| = 880$, (cf. **[4]**).

REFERENCES

1. C. Libis, J. Phillips, and M. Spall, On a certain class of multiplicative magic squares, to appear.
2. W. Benson and O. Jacoby, *New Recreations with Magic Squares*, Dover, New York, NY, 1976.
3. Macalester College Problem of the Week, Nov. 30, 1994.
4. Dame K. Ollerenshaw and Sir H. Bondi, Magic squares of order four, *Phil. Trans. R. Lond.* A 306 (1982) 443–532.

An old Quickie (January/February 1950)

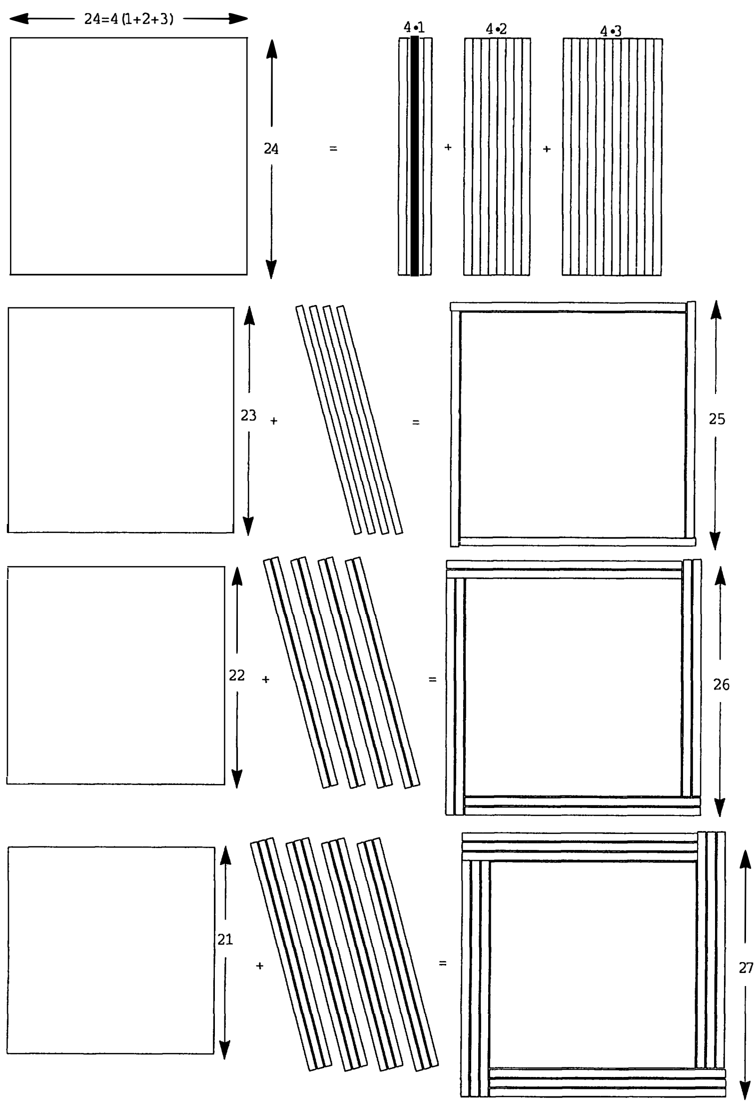
A cube of wood 3 inches on each edge is to be cut into cubes 1 inch on each edge. If, after each cut with a saw, the pieces may be piled in any desired manner before making the next cut, what is the smallest number of different "cuts through the pile" that will accomplish the desired dissection? (*Solution on page 70.*)

Proof Without Words: Pythagorean Runs

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$\dots$$

$$T_n = 1 + 2 + \cdots + n \Rightarrow (4T_n - n)^2 + \cdots + (4T_n)^2 = (4T_n + 1)^2 + \cdots + (4T_n + n)^2$$

For $n = 3$:



—Michael Boardman
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Shortest Shoelaces

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Introduction Halton [1] first studied the question of finding the shortest possible lacing of a shoe. Misiurewicz [2] generalized Halton's original result to handle irregularly placed eyelets. We generalize Halton's result in a different direction by considering lacings that do not necessarily alternate in a regular way.

Halton proved that the American style lacing is the shortest among all possible alternating lacings. However, some common lacings are not covered by Halton's definition. Neither the *ice skater's lacing* (FIGURE 1A) nor the *playground lacing* (FIGURE 1B) is alternating.

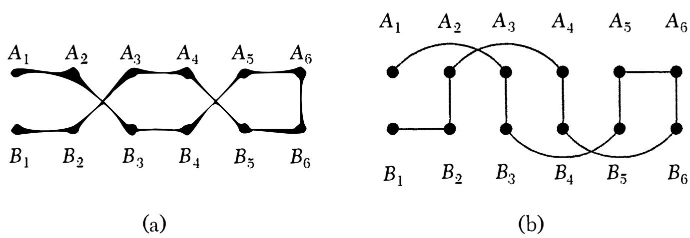

FIGURE 1
(a) ice skater's lacing, (b) playground lacing

Something to avoid in sensible lacings is the occurrence of three consecutive eyelets on the same side. In this case, why bother to use the middle one? This condition is equivalent to requiring that every eyelet (except possibly the first and last) be the endpoint of at least one crossing.

A *lacing* of degree n is an ordering of the set

$$\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}$$

starting with A_1 and ending with B_1, such that no more than two A's (respectively B's) occur in consecutive places. A *bipartite lacing* of degree n is a lacing of degree n in which the A's and B's alternate. Halton's definition is equivalent to our definition of bipartite lacings.

The main result Following Halton, we shall assume that the two rows of eyelets are parallel and that the eyelets in each row are evenly spaced. A simple calculation shows that the ice skater's lacing is significantly shorter than the American style lacing. In fact, the ice skater's lacing is minimal. A modification of Halton's original proof for bipartite lacings [1] demonstrates this fact.

THEOREM. *For any n, L_{IS} is a shortest lacing of degree n.*

The idea of the proof is as follows. Start with a lacing L. Following Halton, create a path P in a rectangular grid so that the path starts in the upper left corner and so that all the segments of P are horizontal, vertical, or diagonal downward and to the right. FIGURE 2 illustrates this transformation in two cases. The dashed line represents L_{IS}, while the solid line represents another particular lacing.

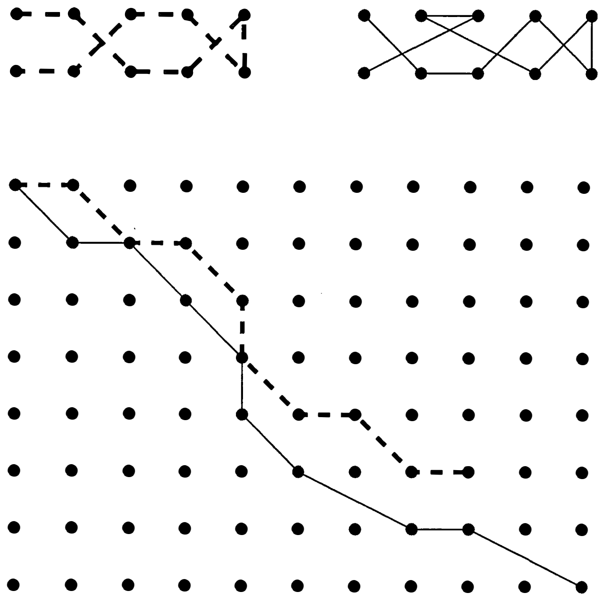

FIGURE 2

At top, the iceskater's lacing and another lacing. At bottom, these lacings are unwound.

We do not know exactly where P ends. The original ice skater's lacing has as many non-crossing steps as possible. Hence P_{IS} ends somewhere above (or possibly in the same row as) the endpoint of P.

Moreover, the horizontal length of P must be at least $2n-2$ units since L connects A_1 to A_n to B_1. The horizontal length of P_{IS} is exactly $2n-2$ units, so we know that P must end to the right of (or possibly in the same column as) the endpoint of P_{IS}.

From this point, slightly technical but straightforward arguments, which we leave as an exercise for the reader, show that P is no shorter than P_{IS}. One way to show this is to lengthen P_{IS} by replacing horizontal segments with diagonal segments until P and P_{IS} end in the same row. Then cancel horizontal and vertical segments of equal length from P_{IS} and P until P_{IS} becomes a straight line. This completes the proof.

Uniqueness For n even, L_{IS} is the *unique* shortest lacing. When n is odd, there are exactly $(n+1)/2$ shortest lacings. They differ only by reordering the horizontal and crossing segments.

Now you know the best way of temporarily relacing your shoe the next time your shoelace breaks!

REFERENCES

1. John H. Halton, The shoelace problem, *Math. Intelligencer* 17 (no. 4) (1995), 36–41.
2. Michal Misiurewicz, Lacing irregular shoes, *Math. Intelligencer* 18 (no. 4) (1996), 32–34.

PROBLEMS

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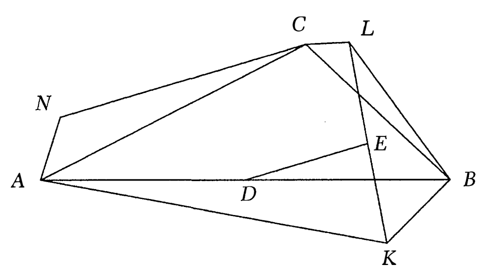
Proposals

To be considered for publication, solutions should be received by July 1, 2000.

1589. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington, Pennsylvania.*

On the sides of $\triangle ABC$, three similar triangles, AKB, BLC, and CNA, are drawn outward. If AB and KL are bisected by D and E respectively, prove that DE is parallel to NC and determine DE/NC.



1590. *Proposed by Constantin P. Niculescu, University of Craiova, Craiova, Romania.*

For given a, $0 < a \leq \pi/2$, determine the minimum value of $\alpha \geq 0$ and the maximum value of $\beta \geq 0$ for which

$$\left(\frac{x}{a}\right)^{\alpha} \leq \frac{\sin x}{\sin a} \leq \left(\frac{x}{a}\right)^{\beta}$$

holds for $0 < x \leq a$.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. *Each solution should begin on a separate sheet containing the solver's name and full address.*

Solutions and new proposals should be mailed to George T. Gilbert, Problems Editor, Department of Mathematics, Box 298900, Texas Christian University, Fort Worth, TX 76129, or mailed electronically (ideally as a LATEX file) to g.gilbert@tcu.edu. *Readers who use e-mail should also provide an e-mail address.*

(This generalizes the well known inequality due to Jordan, which asserts that $2x/\pi \leq \sin x \leq 1$ on $[0, \pi/2]$.)

1591. *Proposed by Western Maryland College Problems Group, Westminster, Maryland.*

We call a 3-tuple (a, b, c) of positive integers a *triangular triple* if

$$T_a + T_b = T_c,$$

where $T_n = n(n+1)/2$ is the nth triangular number. Given an integer k, prove that there are infinitely many triples of distinct triangular triples, (a_i, b_i, c_i) for $i = 1, 2, 3$, such that

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = k^3.$$

1592. *Proposed by Ho-joo Lee, student, Kwangwoon University, Seoul, South Korea.*

Let $ABCDE$ be a cyclic pentagon. Prove that

$$\cot \angle ABC + \cot \angle ACB = \cot \angle AED + \cot \angle ADE$$

if and only if

$$\cot \angle ABD + \cot \angle ADB = \cot \angle AEC + \cot \angle ACE.$$

1593. *Proposed by Jon Florin, Chur, Switzerland.*

Let $(f_n)_{n \geq 1}$ be a sequence of continuous, monotonically increasing functions on the interval [0, 1] such that $f_n(0) = 0$ and $f_n(1) = 1$. Furthermore, assume

$$\sum_{n=1}^{\infty} \max \leq |f_n(x) - x| < \infty.$$

(In particular, $(f_n)_{n \geq 1}$ converges to the identity on [0, 1].)

(a) Must $(f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)_{n \geq 1}$ converge to a continuous function?

(b) Must $(f_1 \circ f_2 \circ \cdots \circ f_{n-1} \circ f_n)_{n \geq 1}$ converge to a continuous function?

Quickies

Answers to the Quickies are on page 70.

Q897. *Proposed by Charles Vanden Eynden, Illinois State University, Normal, Illinois.*

Prove: If a, b, and c are positive integers such that $a | b^c$, then $a | b^a$.

Q898. *Proposed by Mihály Bencze, Braşov, Romania.*

Let n be a positive integer and $f\colon [a, b] \to (a, b)$ be continuous. Prove that there exist distinct $c_1, c_2, \ldots, c_n$ in $[a, b]$ that are in arithmetic progression and satisfy

$$f(c_1) + f(c_2) + \cdots + f(c_n) = c_1 + c_2 + \cdots + c_n.$$

Solutions

An Inequality in a Quadrilateral **February 1999**

1564. *Proposed by Wu Wei Chao, Guang Zhou Normal University, Guang Zhou City, Guang Dong Province, China.*

Let P be the intersection of the diagonals of convex quadrilateral $ABCD$ with $\angle BAC + \angle BDC = 180^\circ$. Suppose that the distance from A to the line BC is less than the distance from D to BC. Show that

$$\left(\frac{AC}{BD}\right)^2 > \frac{AP \cdot CD}{DP \cdot AB}.$$

Solution by Oum Sang-il, Seoul, South Korea.

We first show that $[ABC] \cdot [ACD] > [BCD] \cdot [ABD]$. From

$$\angle ABC + \angle BCD = 360^\circ - (\angle BAD + \angle CDA) < 360^\circ - (\angle BAC + \angle BDC) = 180^\circ$$

and the convexity of $ABCD$, it follows that the distance from C to the line AB is greater than the distance from D to the line AB, hence $[ABC] > [ABD]$. Because the distance from A to the line BC is less than the distance from D to the line BC, we know that $[BCD] > [ABC]$. Then

$$\begin{aligned}
&[ABC] \cdot [ACD] - [BCD] \cdot [ABD] \\
&\quad = [ABC]([ABCD] - [ABC]) - [BCD]([ABCD] - [BCD]) \\
&\quad = ([ABC] - [BCD])([ABCD] - [ABC] - [BCD]) \\
&\quad = ([ABC] - [BCD])([ABD] - [ABC]) > 0.
\end{aligned}$$

We now substitute

$$\begin{aligned}
[ABC] &= \frac{1}{2} \cdot AB \cdot AC \cdot \sin \angle BAC, \\
[ACD] &= \frac{1}{2} \cdot AC \cdot DP \cdot \sin \angle CPD, \\
[BCD] &= \frac{1}{2} \cdot BD \cdot CD \cdot \sin \angle BDC = \frac{1}{2} \cdot BD \cdot CD \cdot \sin \angle BAC, \\
[ABD] &= \frac{1}{2} \cdot BD \cdot AP \cdot \sin \angle APB = \frac{1}{2} \cdot BD \cdot AP \cdot \sin \angle CPD,
\end{aligned}$$

obtaining $AC^2 \cdot AB \cdot DP > BD^2 \cdot CD \cdot AP$. It follows that

$$\left(\frac{AC}{BD}\right)^2 > \frac{AP \cdot CD}{DP \cdot AB}.$$

Also solved by Robin Chapman (United Kingdom), Con Amore Problem Group (Denmark), Daniele Donini (Italy), Robert L. Doucette, Jiro Fukuta (Professor Emeritus, Japan), Victor Y. Kutsenok, Peter Y. Woo, Bilal Yurdakul (Turkey), and the proposer.

Selling Stamps **February 1999**

1565. *Proposed by Joaquín Gómez Rey, I. B. "Luis Buñuel," Alcorcón, Madrid, Spain.*

A philatelist has $(n+1)! - 1$ stamps and decides to sell a portion of them in n steps. In each step he will sell $1/(k+1)$ of his remaining total plus $1/(k+1)$ of one stamp, for $k = 1, \dots, n$. However, these n steps are ordered randomly. Let p_n denote the probability that he does not sell the same number of stamps in two successive steps. Evaluate $\lim_{n\to\infty} p_n$.

I. *Solution by Michael H. Andreoli, Miami Dade Community College (North), Miami, Florida.*

We will show that $\lim_{n\to\infty} p_n = 1/e$.

Let $i \mapsto x_i$ be a random permutation of $\{1,2,\dots,n\}$. On the kth sale, $1/(x_k+1)$ of the remaining total plus $1/(x_k+1)$ of one stamp will be sold. Denote the number of stamps sold on step k by G_k. Then $G_1 = (n+1)!/(x_1+1)$ and $G_k = [(n+1)! - G_1 - G_2 - \cdots - G_{k-1}]/(x_k+1)$ for $k > 1$. A straightforward induction verifies that

$$G_k = \frac{(n+1)!x_1x_2\cdots x_{k-1}}{(x_1+1)(x_2+1)\cdots(x_k+1)}$$

for $k > 1$. From this latter expression, we see that $G_k = G_{k+1}$ if and only if $x_k = x_{k+1} + 1$. Thus, p_n is the probability that, in a random permutation of $\{1,2,\dots,n\}$, i is never immediately followed by $i-1$ for $i = 2,3,\dots,n$.

Let A_i denote the event that i is immediately followed by $i-1$ in a random permutation. We use the inclusion-exclusion formula to calculate $p_n = P(\overline{A_2} \cap \overline{A_3} \cap \cdots \cap \overline{A_n})$. Note first that there are $\binom{n-1}{k}$ k-tuple intersections of the A_i, each of which has probability $(n-k)!/n!$. Inclusion-exclusion then gives

$$\begin{aligned} p_n &= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(n-k)!}{n!} (-1)^k = \sum_{k=0}^{n-1} \frac{n-k}{n\cdot k!}(-1)^k \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} - \frac{1}{n}\sum_{k=1}^{n-1} \frac{(-1)^k}{(k-1)!}. \end{aligned}$$

It follows that $\lim_{n\to\infty} p_n = 1/e$.

II. *Solution by Robin Chapman, University of Exeter, Exeter, United Kingdom.*

Suppose at successive stages the philatelist sells $1/(k+1)$ of his remaining stamps plus $1/(k+1)$ of a stamp, and $1/(l+1)$ of his remaining stamps plus $1/(l+1)$ of a stamp. If before the former of these stages he has $r-1$ stamps, then he sells $r/(k+1)$ of the stamps in the former stage, leaving him $kr/(k+1) - 1$ stamps and he sells $kr/[(k+1)(l+1)]$ in the latter stage. He sells the same number of stamps in both of these stages if and only if $k = l+1$. Let $(k_1, k_2, \dots, k_n)$ be the random sequence of k's. These form a permutation of $1, 2, \dots, n$ and the philatelist never sells the same number of stamps on successive days if and only if $k_{i+1} \neq k_i - 1$ for $1 \le i < n$. Let S_n denote the set of permutations of the set $\{1, 2, \dots, n\}$ considered as the set of all ways $(k_1, k_2, \dots, k_n)$ of writing the numbers $1, 2, \dots, n$ in order. Let A_n be the set of $(k_1, k_2, \dots, k_n) \in S_n$ with no $k_{i+1} = k_i - 1$ and let $a_n = |A_n|$. Then $p_n = a_n/n!$.

We aim to provide a recurrence for a_n. Given $n \ge 1$ if we remove the occurrence of n from an element $(k_1, k_2, \dots, k_n)$ of A_n we either get an element of A_{n-1} or an element $l_1, l_2, \dots, l_{n-1}$ of S_{n-1} where exactly one $l_{i+1} = l_i - 1$. Let B_n be the set of elements $(k_1, k_2, \dots, k_n) \in S_n$ with exactly one instance where $k_{i+1} = k_i - 1$ and let $b_n = |B_n|$. Given an element of A_{n-1} we can create an element of A_n by inserting n in exactly $n-1$ ways: by inserting n before any of the numbers $1, 2, \dots, n-2$ or at the end. Given an element of B_{n-1} we can create an element of A_n by inserting n in exactly one way: between the two adjacent numbers which decrease by 1. Hence

$$a_n = (n-1)a_{n-1} + b_{n-1}.$$

We now need to count B_n. Given $(l_1, l_2, \ldots, l_n) \in B_n$ consider the unique $l = l_i$ with $l_{i+1} = l - 1$. If we remove l and reduce each l_j in the permutation with $l_j > l$ by 1, we get an element of A_{n-1}. Given an element in A_{n-1} and $l \in \{2, 3, \ldots, n\}$ we can reverse the procedure to get an element of B_n. Hence for $n \geq 1$ we have $b_n = (n-1)a_{n-1}$. Thus

$$a_n = (n-1)a_{n-1} + (n-2)a_{n-2}.$$

The recurrence is valid for all $n \geq 2$ with the convention that $a_0 = a_1 = 1$. To solve this recurrence set $r_n = na_n$. Then

$$r_n = nr_{n-1} + nr_{n-2}.$$

Now set $s_n = r_n - (n+1)r_{n-1}$ for $n \geq 1$. Then $s_n = -s_{n-1}$ for $n \geq 2$ and so

$$s_n = (-1)^{n-1}s_1 = (-1)^{n-1}.$$

Now set $t_n = r_n/(n+1)!$. We get

$$t_n - t_{n-1} = \frac{r_n - (n+1)r_{n-1}}{(n+1)!} = \frac{(-1)^{n-1}}{(n+1)!}$$

and so

$$t_n = t_0 + \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(j+1)!} = \sum_{j=2}^{n+1} \frac{(-1)^j}{j!} = \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}.$$

Thus

$$p_n = a_n/n! = r_n/(n \cdot n!) = (n+1)t_n/n = \frac{n+1}{n} \sum_{j=0}^{n+1} \frac{(-1)^j}{j!}$$

and $p_n \to 1/e$ as $n \to \infty$ as required.

Also solved by Jean Bogaert (Belgium), Con Amore Problem Group (Denmark), Daniele Donini (Italy), Robert L. Doucette, Kathleen E. Lewis, Michael Reid, Oum Sang-il (South Korea), and the proposer.

Area and Perimeter Ratios of Inscribed Rectangles — February 1999

1566. *Proposed by Stephen G. Penrice, Morristown, New Jersey.*

Let circle C circumscribe (nondegenerate) rectangle R. Let α be the ratio of the area of C to the area of R, and let β be the ratio of the circumference of C to the perimeter of R. Show that α and β cannot both be algebraic.

Solution by Con Amore Problem Group, Copenhagen, Denmark.

Let C have radius r and let R have sides x and y. Then we have

$$\alpha = \frac{\pi r^2}{xy} \quad \text{and} \quad \beta = \frac{2\pi r}{2x + 2y} = \frac{\pi r}{x+y},$$

and, therefore,

$$x+y=\frac{\pi r}{\beta} \quad \text{and} \quad xy=\frac{\pi r^2}{\alpha}.$$

It follows that

$$4r^2=x^2+y^2=(x+y)^2-2xy=\frac{\pi^2 r^2}{\beta^2}-\frac{2\pi r^2}{\alpha},$$

so that

$$4=\frac{\pi^2}{\beta^2}-\frac{2\pi}{\alpha} \quad \text{or} \quad 4\alpha\beta^2=\alpha\pi^2-2\beta^2\pi.$$

We have found that π satisfies the equation $\alpha z^2-2\beta^2 z-4\alpha\beta^2=0$. If both α and β were algebraic, then π would also be algebraic, which it is not.

Also solved by Roy Barbara (Lebanon), Robin Chapman (United Kingdom), Ivko Dimitric, Daniele Donini (Italy), Hans Kappus (Switzerland), Murray S. Klamkin (Canada), Kee-Wai Lau (China), Can A. Minh (graduate student), Christopher Pilman and Daniel Bombeck, Michael Reid, Heinz-Jürgen Seiffert (Germany), Southwest Missouri State University Problem Solving Group, Monte J. Zerger, and the proposer. There was one incorrect solution.

The Smith Normal Form of a Matrix **February 1999**

1567. *Dennis Spellman, Philadelphia, Pennsylvania, and William P. Wardlaw, United States Naval Academy, Annapolis, Maryland.*

Find the Smith normal form over the integers of the $n\times n$ matrix A with entries $a_{ij}=j^i$.

(The Smith normal form of an integral matrix A is a diagonal matrix D that is obtained through applying to A a sequence of elementary row and column operations with integral matrices of determinant ± 1. It is defined to be the unique such matrix whose diagonal entries d_{ii} are nonnegative integers satisfying (i) $d_{ii}\neq 0$ if and only if $i\leq \text{rank}(A)$ and (ii) d_{ii} divides $d_{(i+1)(i+1)}$ for $1\leq i<\text{rank}(A)$.)

Solution by John H. Smith, Boston College, Chestnut Hill, Massachusetts.

Let $D(k)$ be the $k\times k$ diagonal matrix with diagonal entries $d_{ii}=i!$. We show that there are unimodular matrices U and L which are upper and lower triangular, respectively, such that $LAU=D(n)$, hence $D(n)$ is the Smith normal form of A.

Let

$$A(k)=\begin{pmatrix} D(k) & 0 \\ 0 & C(k)\end{pmatrix} \quad \text{and}$$

$$C(k)=\begin{pmatrix} \frac{(k+1)!}{0!} & (k+1)\frac{(k+1)!}{0!} & \cdots & (k+1)^{n-k-1}\frac{(k+1)!}{0!} \\ \frac{(k+2)!}{1!} & (k+2)\frac{(k+2)!}{1!} & \cdots & (k+2)^{n-k-1}\frac{(k+2)!}{1!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n!}{(n-k-1)!} & n\frac{n!}{(n-k-1)!} & \cdots & n^{n-k-1}\frac{n!}{(n-k-1)!}\end{pmatrix}.$$

Note that $A(0)=A$ and $A(n)=D(n)$, so it will suffice to show that there are unimodular matrices $U(k)$ and $L(k)$ which are upper and lower triangular, respectively, such that $L(k)A(k)U(k)=A(k+1)$. We describe these matrices by describing their action on the columns and rows of $C(k)$. First, $U(k)$ subtracts $k+1$ times the next-to-last column from the last, then $k+1$ times the third-to-last column from the next-to-last, and so forth, ending by subtracting $k+1$ times the first column from the second. This produces

$$\begin{pmatrix} \frac{(k+1)!}{0!} & 0 & \cdots & 0 \\ \frac{(k+2)!}{1!} & \frac{(k+2)!}{0!} & \cdots & (k+2)^{n-k-2}\frac{(k+2)!}{0!} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{n!}{(n-k-1)!} & \frac{n!}{(n-k-2)!} & \cdots & n^{n-k-2}\frac{n!}{(n-k-2)!} \end{pmatrix}.$$

Because $(k+1)!$ divides the product of any $k+1$ consecutive integers, we may subtract appropriate multiples of the first row from lower ones to make the rest of the first column zero, yielding $\begin{pmatrix} (k+1)! & 0 \\ 0 & C(k+1) \end{pmatrix}$, which, combined with $D(k)$, gives $A(k+1)$.

Also solved by Michel Bataille (France), David Callan, Robin Chapman (United Kingdom), Con Amore Problem Group (Denmark), Daniele Donini (Italy), Robert L. Doucette, Reiner Martin, and the proposer.

Groups with Restricted Intersections of Subgroups **February 1999**

1568. *Proposed by Emre Alkan, student, University of Wisconsin, Madison, Wisconsin.*

Determine which finite, nonsimple groups G satisfy the following: If H and K are subgroups of G, then either (i) $H \subset K$, (ii) $K \subset H$, or (iii) $H \cap K = \{e\}$.

Solution by Joel D. Haywood, Macon State College, Macon, Georgia.

We show that a finite group G, whether simple or not, satisfies the given condition, which we will refer to as C, if and only if for some distinct primes p and q, G is (a) cyclic of order a nonnegative power of p, (b) noncyclic of order p^2, (c) of order pq, or (d) a semi-direct product $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_q$ with no subgroup of order pq.

We first prove sufficiency. A cyclic group has a unique subgroup of every order dividing the order of the group and, for groups of prime power order, those subgroups are totally ordered by inclusion, which proves sufficiency for (a). By Lagrange's Theorem, any two subgroups of prime order are either equal or have trivial intersection, which proves sufficiency for (b) and (c). In (d), the Sylow p-subgroup is normal, hence unique. All subgroups of order p are contained in this Sylow p-subgroup. All other pairs of nontrivial, proper subgroups have trivial intersection, which proves sufficiency for (d).

We turn to necessity.

Because any subgroup of H is a subgroup of G, if H does not satisfy C, then G does not either. Therefore, if G satisfies C and H is a subgroup of G, then H satisfies C.

We next prove that if G satisfies C and is a p-group, then G is cyclic or $o(G) = p^2$. Suppose G is nonabelian. Then there exist elements a and b of G such that $ab \neq ba$. Thus, the centralizers of a and b are subgroups of G such that neither one contains the other. Thus, by C, their intersection is trivial. Because $Z(G)$ is in their intersection, $Z(G)=\{e\}$. Because G is a p-group, $Z(G)\neq\{e\}$. Thus, G is abelian. If G is not cyclic, then G is the direct product of two subgroups H and K. Let H' be a proper subgroup of H. Then, $H'\times K$ and H are subgroups of G such that neither contains the other. Thus, by C, their intersection is trivial. Because H' is in their intersection, H' is trivial. Thus, H is a group with no nontrivial proper subgroups, and, thus, has prime order. Similarly, K has prime order. Thus, $o(G)=o(H)\cdot o(K)=p^2$.

Our next step is to prove that if G satisfies C and has order divisible by two distinct primes, then $o(G)$ is divisible by only those two primes and contains a normal Sylow subgroup. Let P be a Sylow p-subgroup of G and Q be a Sylow q-subgroup of G such that $o(P)=p^n$ and $o(Q)=q^m$. Then, $N(N(P))=N(P)$ and $N(N(Q))=N(Q)$. Thus, there are $[G\colon N(N(P))]=[G\colon N(P)]$ distinct conjugates of $N(P)$ in G. Because all such conjugates have the same order, then, by C, any two of them are equal or have trivial intersection. Thus, those conjugates contain:

$$[G\colon N(P)]\cdot(O(N(P))-1)+1=o(G)-[G\colon N(P)]+1$$

distinct elements of G, and similarly for $N(Q)$. Now, suppose the intersection of any conjugate of $N(P)$ and any conjugate of $N(Q)$ is trivial. Then, all these conjugates contain $2o(G)-[G\colon N(P)]-[G\colon N(Q)]+1$ distinct elements of G. Thus,

$$1\leq[G\colon N(P)]+[G\colon N(Q)]-o(G)=o(G)(1/o(N(P))+1/o(N(Q))-1).$$

Because both $o(N(P))$ and $o(N(Q))$ are integers, at least one of them is equal to 1, which is a contradiction. Thus, some conjugate of $N(Q)$ meets $N(P)$ nontrivially.

Without loss of generality, by C, some conjugate of $N(Q)$ is a subset of $N(P)$. We may choose Q such that $N(Q)$ is a subset of $N(P)$. Because P is normal in $N(P)$, $\langle P,Q\rangle$ has order p^nq^m. Now, suppose another prime r divides $o(G)$. If the Sylow r-subgroup has order r^k, then the same argument implies G has a subgroup of order p^nr^k that contains P, a contradiction of C. Thus, $o(G)=p^nq^m$ and P is normal in G.

It remains to show that if $o(G)=p^nq^m$ with P is normal in G, then G is either of order pq or a semi-direct product $(\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_q$ with no subgroup of order pq. Suppose H is a subgroup of G of order pq. Then, there exist elements a and b of H such that $o(a)=p$ and $o(b)=q$. Thus, a is contained in a Sylow p-subgroup of G and b is contained in a Sylow q-subgroup of G. Thus, by C, H contains both those Sylow subgroups and $o(G)=pq$.

Now assume that G has no subgroup of order pq. We know that P satisfies C so must be either cyclic or (isomorphic to) $\mathbb{Z}_p\times\mathbb{Z}_p$. Suppose P is cyclic. Then P contains a unique subgroup K of order p. Because P is the unique Sylow p-subgroup of G, K is the unique subgroup of order p and hence normal in G. Let $b\in G$ have order q. Because K is normal in G, the subgroup $\langle K,b\rangle$ has order pq, which is a contradiction. Therefore, P is $\mathbb{Z}_p\times\mathbb{Z}_p$. Now if $b\in G$ has order q, then $\langle P,b\rangle$ intersects a Sylow q-subgroup of G containing b nontrivially, hence must contain this Sylow q-subgroup. It follows that G is a semi-direct product $(\mathbb{Z}_p\times\mathbb{Z}_p)\rtimes\mathbb{Z}_q$ with no subgroup of order pq.

The proof of necessity is now complete.

Also solved by Michel Bataille (France), Robin Chapman (United Kingdom), Daniele Donini (Italy), and Kandasamy Muthuvel. There were two incorrect solutions.

Answers

Solutions to the Quickies on page 63.

A897. If the claim is false, then there exist a prime p and a positive integer m such that p^m divides a but not b^a. Now $p|b$ because $a|b^c$. Thus $p^a|b^a$, so $a<m$. Then $2^a \le p^a < p^m \le a$, a contradiction.

A898. Because $f(a)>a$, $f(b)<b$, and f is continuous, for $\delta>0$ sufficiently small, $f(a+k\delta)>a+k\delta$ and $f(b-k\delta)<b-k\delta$ for $k=0,1,\ldots,n-1$. Thus, the continuous function

$$g(x)=f(x)+f(x+\delta)+f(x+2\delta)+\cdots+f(x+(n-1)\delta)$$
$$-(x+(x+\delta)+(x+2\delta)+\cdots+(x+(n-1)\delta))$$

satisfies $g(a)>a$ and $g(b-(n-1)\delta)<b-(n-1)\delta$. It follows that there exists $c\in[a,b-(n-1)\delta]$ such that $g(c)=c$. Setting $c_k=c+(k-1)\delta$ for $k=1,2,\ldots,n$ completes the proof.

Answer to old Quickie, page 58

Six cuts are required; for not matter how the cutting is done, the 6 faces of the central cube must result from separate cuts. The job may be done without any piling at all.

An old Problem (March 1940)

Find such a four-digit number that when 385604 is written at its right the result is a perfect square. (*Proposed by V. Thébault, Le Mans, France.*)

REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Mullen, William, Eureka! Rare Archimedes manuscript finds its way to Field, *Chicago Tribune* (3 November 1999). *The Archimedes Palimpsest*, videotape, Walters Art Gallery (600 North Charles St., Baltimore, MD 21201), 1999; 30 min, \$14.95. Stein, Sherman, *Archimedes: What Did He Do Besides Cry Eureka?*, Mathematical Association of America, 1999; x + 155 pp, \$24.95 (P). ISBN 0-88385-328-0.

The Archimedes palimpsest, which contains the only version of his *Method Treating of Mechanical Problems* (*The Method*, for short), was acquired by an anonymous collector in October 1998 for \$2 million. It was on display July–August at the Walters Art Gallery in Baltimore and traveled to the Field Museum in Chicago for November–December 1999. By the time you read this, it will have returned to the Walters, where it will be restored over the next two years before being digitized and transcribed.

The book itself is in extremely poor condition, with charred edges and severe mold on its pages. At Chicago, it was opened to a page on which the observer was (necessarily) advised by the exhibitors to look in the margin to find what appeared to be the only visible traces of the Archimedes manuscript below the overlaid church prayers. Very little more was visible on a separated page under ultraviolet light. The transcription in 1906 by J.L. Heiberg equipped only with a magnifying glass would seem impossible today. A photograph that he made of one page (reproduced on p. 30 of Stein's book) suggests that the text was much clearer then, and hence the book must have suffered far greater damage in the past century than in the preceding six hundred years. The cover of Stein's book shows a digitally enhanced photograph of a page, and the text appears with remarkable clarity—a sign that the wait for the restoration and transcription will be worthwhile, though it is unlikely that there will be any great new revelations.

Meanwhile, the Walters has produced an exciting video about the palimpsest's history, with further details at `http://www.thewalters.org/archimedes/frame.html`. The video avoids questions of ownership; the Greek Patriarch says the palimpsest was stolen from his library and tried to stop the auction sale, and the video shows the back of the second-last bidder—the Greek consul, at \$1.9 million—without mention. The video mentions that one page is at Cambridge University, apparently torn out and stolen in 1846 by K. von Tischendorf, whose mention of the palimpsest was crucial to Heiberg's discoveries.

The video also gives no hint whatever about the mathematical contents; for that you may turn to Stein's book for an exposition understandable at the high-school level. Stein devotes a chapter each to the life of Archimedes and to the palimpsest itself but the remaining nine chapters to Archimedes' results in *The Method* and other works (though Stein doesn't always make clear which work a result comes from). Stein uses affine transformations to derive a few key facts, an anachronism that makes the proofs easier but obscures some of what Archimedes did.

Thomas Heath succinctly summarizes *The Method* in his *A History of Greek Mathematics*, Vol. 2, 27–34; New York, Dover. The text of it appears in translation in Heath's *The Works of Archimedes* (New York: Dover, 1953; reprinted in Encyclopædia Britannica's

Great Books of the Western World, Vol. 10 (not 11, as Stein cites), 559–592) and in E.J. Dijksterhuis's *Archimedes* (Princeton, NJ: Princeton University Press, 1987). Heiberg's Greek transcription is in "Eine neue Archimedeshandschrift," *Hermes* 42 (1907) 235–303.

Maz'ya, Vladimire, and Tatyana Shaposhnikova, *Jacques Hadamard, a Universal Mathematician*, American Mathematical Society and London Mathematical Society, 1998; reprinted with corrections 1999, \$49 (P). xxv + 574 pp. ISBN 0–8218–1923–2.

This is a beautifully illustrated, well-designed, and thorough book about Hadamard (1865–1963). About 60% is devoted to his life, and the rest recounts his work in a variety of fields: analytic function theory, number theory, analytical mechanics, calculus of variations, mathematical physics, and partial differential equations. The descriptions of the mathematical work are technical but with a low density of equations and a great deal of prose explanation and motivation. One could hardly wish for a more readable and enticing volume.

Barbeau, Edward J., *Mathematical Fallacies, Flaws, and Flimflam*, Mathematical Association of America, 1999; xvi + 167 pp, \$23.95 (P). ISBN 0–88385–529–1.

This intriguing book collects faulty mathematical arguments from the column of the same title in *College Mathematics Journal* for the past 11 years. Here the fallacies are categorized by area. The result is a rich collection of conundrums; use them to amaze your friends, confound your colleagues, and mystify your students, till they beg for the explanation.

Ross, Sheldon M., *An Introduction to Mathematical Finance*, Cambridge University Press, 1999; xv + 184 pp, \$34.95. ISBN 0–521–77043–2.

This book is a lean and carefully streamlined approach to the Black-Scholes call option formula by a master of exposition about probability. The dust jacket claims correctly that "No other text presents such sophisticated topics in a mathematically accurate but accessible way." Integrals appear only on p. 55, and derivatives are almost as scarce; but this is a book for students with the maturity of a year of calculus behind them.

Francis, Richard L., 21 problems for the twenty-first century, *Consortium: The Newsletter of the Consortium for Mathematics and Its Applications (COMAP)* No. 71 (Fall 1999) 7–10.

Hilbert's famous address to the 1900 International Congress of Mathematicians set out 20 problems for mathematicians of this century. Francis gives 21 for the next, based on a worldwide survey of mathematicians. The problem statements are very succinct (they almost fit on one page), excluding, as Francis admits, problems more difficult to state, such as the Riemann Hypothesis and the Poincaré Conjecture—after all, his article is directed to high-school mathematics teachers. Hence his problems are predominantly from elementary number theory. Still, he enunciates challenging problems that respond in understandable terms—and may inspire—students who wonder what is still unknown in mathematics.

Steiner, Ray (`steiner@math.gbsu.edu`), Possible breakthrough in Catalan's conjecture, Usenet newsgroup `sci.math` 15 December 1999.

Francis's Problem 14 (see above review) is Catalan's Problem: "Are 8 and 9 the only exact powers that are consecutive integers?" It may be solved *before* the twenty-first century (i.e., by 1/1/01). The conjecture amounts to $|x^p - y^q| = 1$ has no nontrivial integer solutions for odd primes p and q, $p < q$. New results show that $10^7 < p < 7.15 \times 10^{11}$ and $q < 7.78 \times 10^{16}$ and that p and q must form a *double Wieferich pair*, i.e., $p^{q-1} \equiv 1 \pmod{q^2}$ and $q^{p-1} \equiv 1 \pmod{p^2}$. These and other limitations suggest that a distributed computation could search all 10^{20} possible pairs "quite rapidly." (Thanks to Darrah Chavey of Beloit College.)

NEWS AND LETTERS

60th Annual William Lowell Putnam Mathematical Competition

Editor's note: Additional Putnam solutions will appear later in the *American Mathematical Monthly.*

A-1 Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that, for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1, \\ 3x + 2 & \text{if } -1 \leq x \leq 0, \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

Solution. The functions $f(x) = 3(x + 1)/2$, $g(x) = 5x/2$, and $h(x) = -x + 1/2$ satisfy the conditions. To find these, we observe that the function is piecewise linear, with slope changes at $x = -1$ and $x = 0$. Thus, we are led to test functions of the form $f(x) = A(x + 1)$, $g(x) = Bx$, and $h(x) = mx + b$, for constants A, B, m, and b, yet to be determined. It then follows that

$$\begin{aligned} -A(x+1) + Bx + mx + b &= -1, && \text{if } x < -1; \\ A(x+1) + Bx + mx + b &= 3x + 2, && \text{if } -1 \leq x \leq 0; \\ A(x+1) - Bx + mx + b &= -2x + 2, && \text{if } x > 0. \end{aligned}$$

Equating the x-coefficients in each case yields

$$-A + B + m = 0 \quad (1)$$

$$A + B + m = 3 \quad (2)$$

$$A - B + m = -2 \quad (3)$$

From (1) and (3) we get $m = -1$; then, from (2) and (3), we get $A = 3/2$ and $B = 5/2$. Equating the constant terms in each case yields $-A + b = -1$ and $A + b = 2$. These are simultaneously satisfied by $b = 1/2$.

A-2 Let $p(x)$ be a polynomial that is non-negative for all x. Prove that, for some k, there are polynomials $f_1(x), \dots, f_k(x)$ such that $p(x) = \sum_{j=1}^{k} (f_j(x))^2$.

Solution. We induct on the degree of p. If the degree of p is 0, it is a non-negative constant and therefore is the square of a constant polynomial. Assume the degree of p is greater than 0 and that the result if true for polynomials of degree less than the degree of p. Let p have an absolute minimum at $x = x_0$ (there must be an absolute minimum because p is a polynomial and $p \geq 0$). Then $p(x) = p(x_0) + (x - x_0)^2 q(x)$, with $q(x)$ non-negative for all x. By the induction hypothesis, $q(x) = \sum_j (f_j(x))^2$, so that $p(x) = \left(\sqrt{p(x_0)}\right)^2 + \sum_j ((x - x_0) f_j(x))^2$.

A-3 Consider the power series expansion $\dfrac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} a_n x^n$. Prove that, for each integer $n \geq 0$, there is an integer m such that $a_n^2 + a_{n+1}^2 = a_m$.

Solution. We have

$$\frac{1}{1-2x-x^2} = \frac{1}{(1-(1+\sqrt{2})x)(1-(1-\sqrt{2})x)} = \frac{(1+\sqrt{2})/2\sqrt{2}}{1-(1+\sqrt{2})x} - \frac{(1-\sqrt{2})/2\sqrt{2}}{1-(1-\sqrt{2})x}$$
$$= \frac{1+\sqrt{2}}{2\sqrt{2}} \sum_{n=0}^{\infty} \left(1+\sqrt{2}\right)^n x^n - \frac{1-\sqrt{2}}{2\sqrt{2}} \sum_{n=0}^{\infty} \left(1-\sqrt{2}\right)^n x^n.$$

Hence

$$a_n = \frac{(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}}{2\sqrt{2}}. \qquad (*)$$

Straightforward algebraic manipulation yields

$$a_n^2 + a_{n+1}^2 = \frac{(1+\sqrt{2})^{2n+3} - (1-\sqrt{2})^{2n+3}}{2\sqrt{2}} = a_{2n+2}\,.$$

Note: Equation $(*)$ can also be deduced from the fact that $a_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n$, where $\lambda_1, \lambda_2 = 1 \pm \sqrt{2}$, the roots of $x^2 - 2x - 1$, using $a_0 = 1$, $a_1 = 2$.

A-4 Sum the series $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n}{3^m(n\,3^m + m\,3^n)}$.

Solution. The double series is

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m^2 n^2}{m\,3^n}\left(\frac{1}{n\,3^m} - \frac{1}{n\,3^m + m\,3^n}\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m\,n}{3^m\,3^n} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m\,n^2}{3^n\,(n\,3^m + m\,3^n)}.$$

The latter series is just S, with m and n switched. Thus

$$2S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{m\,n}{3^m\,3^n} = \left(\sum_{m=1}^{\infty} \frac{m}{3^m}\right)^2, \quad \text{and hence} \quad S = \frac{1}{2}\left(\sum_{m=1}^{\infty} \frac{m}{3^m}\right)^2.$$

Using the identity $\displaystyle\sum_{m=0}^{\infty} m x^m = x\frac{d}{dx}\left(\sum_{m=0}^{\infty} x^m\right) = x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$, we get $2S = (3/4)^2$. Hence $S = 9/32$.

A-5 Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)| \le C \int_{-1}^{1} |p(x)|\, dx .$$

Solution. Let $n = 1999$. Consider polynomials $p(x) = \sum_{m=0}^{n} a_m x^m$ such that $\sum_{m=0}^{n} a_m^2 = 1$. Then $\int_{-1}^{1} |p(x)|\, dx$ is a continuous function on this closed and bounded set. Hence it assumes a minimum, μ. The minimum is positive, for if $\int_{-1}^{1} |p(x)|\, dx = 0$, then $p(x)$ is identically 0. For arbitrary coefficients $a_0, a_1, a_2, \ldots, a_m$, let $c = \left(\sum_{m=0}^{n} a_m^2\right)^{1/2}$. Then, by the preceding work, $\int_{-1}^{1} \left|\frac{1}{c}p(x)\right| dx \le \mu$. Hence

$$\int_{-1}^{1} |p(x)|\, dx \ge \mu \left(\sum_{m=0}^{n} a_m^2\right)^{1/2} \ge \mu|a_0| = \mu|p(0)|.$$

This is the desired inequality, with $C = 1/\mu$.

A-6 The sequence $(a_n)_{n\ge 1}$ is defined by $a_1 = 1, a_2 = 2, a_3 = 24$, and, for $n \ge 4$,

$$a_n = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}} .$$

Show that, for all n, a_n is an integer multiple of n.

Solution. Let $A_n = a_n/a_{n-1}$. Then

$$A_n = \frac{a_n}{a_{n-1}} = \frac{6a_{n-1}^2 a_{n-3} - 8a_{n-1}a_{n-2}^2}{a_{n-1}a_{n-2}a_{n-3}} = 6\frac{a_{n-1}}{a_{n-2}} - 8\frac{a_{n-2}}{a_{n-3}} = 6A_{n-1} - 8A_{n-2}.$$

The solution to this linear recurrence with $A_2 = 2$ and $A_3 = 12$ is $A_n = 4^{n-1} - 2^{n-1}$ for $n \ge 4$, as may be verified by mathematical induction. (The formula is easily derived by noting that 2 and 4 are the roots of $x^2 - 6x + 8 = 0$ and so A_n is expressible in the form $c_1 2^n + c_2 4^n$.) This establishes that all the A's and a's are nonzero. Then

$$a_n = a_1 A_2 A_3 \cdots A_n = (4-2)(4^2 - 2^2)\cdots(4^{n-1} - 2^{n-1}).$$

We will prove that n divides a_n by showing that each prime power p^s that divides n also divides a_n. If $p = 2$, the power of 2 that divides a_n is

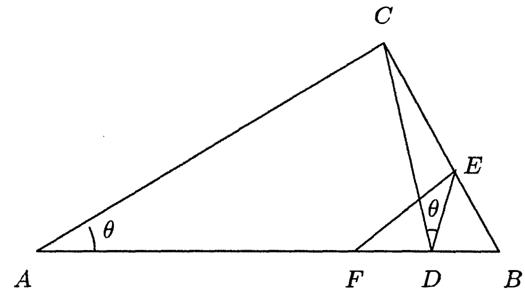
$$2^{1+2+\cdots+(n-1)} = 2^{(n^2-n)/2}.$$

Since $s \le \log_2 n < (n^2 - n)/2$, this shows that 2^s divides a_n. For odd p, Fermat's little theorem implies

$$4^{k(p-1)} - 2^{k(p-1)} \equiv 1 - 1 \equiv 0 \pmod{p}$$

for all integers k. Then the power of p is at least the number of positive k such that $k(p-1) \le n-1$. Again, it is clear that $s \le \log_p(n) < \left\lfloor \frac{n-1}{p-1} \right\rfloor$. This proves that p^s divides a_n.

B-1 Right triangle ABC has right angle at C and $\angle BAC = \theta$; the point D is chosen on AB so that $|AC| = |AD| = 1$; the point E is chosen on BC so that $\angle CDE = \theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim\limits_{\theta\to 0} |EF|$. [Here, $|PQ|$ denotes the length of the line segment PQ.]



Solution. Since $|AC| = |AD|$, we have $\angle ADC = \pi/2 - \theta/2$, and so $\angle BDE = \pi/2 - \theta/2$. Then, since $\angle ABC = \pi/2 - \theta$, we see that $\angle BED = 3\theta/2$. Using the law of sines in $\triangle BED$, we get

$$|BE| = |DB|\,\frac{\sin(\pi/2-\theta)}{\sin(3\theta/2)} = (\sec\theta - 1)\,\frac{\cos\theta}{\sin(3\theta/2)}.$$

Then

$$|EF| = \frac{|EF|}{|AC|} = \frac{|BE|}{|BC|} = \frac{\sec\theta - 1}{\tan\theta}\,\frac{\cos\theta}{\sin(3\theta/2)} = \frac{\sin(\theta/2)}{\sin(3\theta/2)}\,\frac{\cos\theta}{\cos(\theta/2)}.$$

Hence $\lim\limits_{\theta\to 0} |EF| = \dfrac{1}{3}$.

B-2 Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots, then it must have n distinct roots. [The roots may be either real or complex.]

Solution. Let r be a root of $P(x)$ of multiplicity $m \geq 2$. Then the multiplicity of $P''(x)$ at r is $m-2$, and $Q(x)$ must have a double root at r, so $Q(x) = \dfrac{(x-r)^2}{n(n-1)}$. Writing

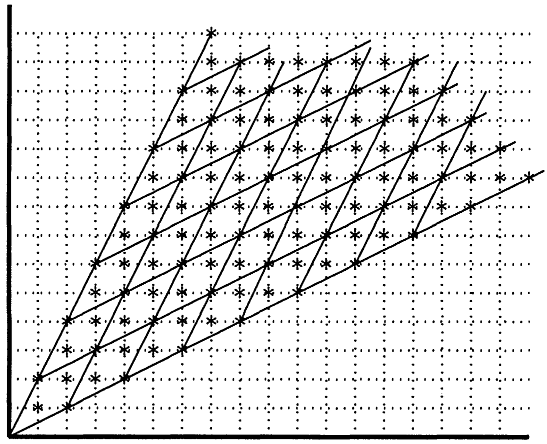
$$P(x) = a_m(x-r)^m + \cdots + (x-r)^n,$$

we obtain $\dfrac{m(m-1)}{n(n-1)}a_m = a_m$, with $a_m \neq 0$, so $m = n$, contrary to hypothesis. Thus $P(x)$ has all roots distinct.

B-3 Let $A = \{(x,y) : 0 \leq x, y < 1\}$. For $(x,y) \in A$, let $S(x,y) = \sum\limits_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n$, where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{\substack{(x,y)\to(1,1)\\(x,y)\in A}} \left(1 - xy^2\right)\left(1 - x^2y\right) S(x,y) .$$

Solution. The exponents (m, n) in the sum correspond to the spots marked below in the first quadrant:



We have grouped these points into congruent "parallelograms," demonstrating that they are all uniquely expressible in the form

$$k(1,2) + \ell(2,1) + (a,b),$$

where k and ℓ are nonnegative integers, and (a,b) is one of $\{(0,0), (1,1), (2,2)\}$. Note that $(0,0)$ is also of this form, and that $(0,0)$ is the only point not included in our sum. Then

$$1 + S(x,y) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} x^{k+2\ell}y^{2k+\ell}(1 + xy + x^2y^2) = (1 + xy + x^2y^2)\sum_{k=0}^{\infty}(xy^2)^k\sum_{\ell=0}^{\infty}(x^2y)^\ell$$
$$= (1 + xy + x^2y^2)(1 - xy^2)^{-1}(1 - x^2y)^{-1}.$$

Thus $(1 - xy^2)(1 - x^2y)S(x,y) = 1 + xy + x^2y^2 - (1 - xy^2)(1 - x^2y)$. The limit of the left side as (x,y) approaches $(1,1)$ is found by evaluating the right side at $x = y = 1$. The limit is then 3.

Note: For general rational lower and upper bounds r/s and p/q respectively, the same argument shows that the analogous limit is $qr - ps$.)

B-4 Let f be a real function with a continuous third derivative such that $f(x)$, $f'(x)$, $f''(x)$, and $f'''(x)$ are positive for all x. Suppose that $f'''(x) \le f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.

Solution. Fix x. By Taylor's theorem, for any $t > 0$,

$$f(x-t) = f(x) - f'(x)t + \frac{f''(x-s)}{2}t^2$$

for some s with $0 < s < t$. Since $f''(x-s) < f''(x)$ and $f(x-t) > 0$, we have

$$f(x) - f'(x)t + \frac{f''(x)}{2}t^2 > 0$$

for all $t > 0$. For $t = f'(x)/f''(x)$, this yields, for all x, $(f'(x))^2 < 2f(x)f''(x)$.

Similarly, for any $t > 0$,

$$f'(x-t) = f'(x) - f''(x)t + \frac{f'''(x-s)}{2}t^2,$$

where $0 < s < t$. Since $f'''(x-s) \le f(x-s) < f(x)$ and $f'(x-t) > 0$, we have, for all $t > 0$,

$$f'(x) - f''(x)t + \frac{f(x)}{2}t^2 > 0.$$

For $t = f''(x)/f(x)$, this yields $(f''(x))^2 < 2f(x)f'(x)$ for all x. Thus, for all x,

$$(f'(x))^4 < 4(f(x))^2(f''(x))^2 < 8(f(x))^3 f'(x),$$

whence $f'(x) < 2f(x)$.

B-5 For an integer $n \ge 3$, let $\theta = 2\pi/n$. Evaluate the determinant of the $n \times n$ matrix $I + A$, where I is the $n \times n$ identity matrix and $A = (a_{jk})$ has entries $a_{jk} = \cos(j\theta + k\theta)$ for all j, k.

Solution. Observe that $\alpha = e^{i\theta} = e^{2\pi i/n}$ is an n-th root of unity, and that $\cos(j\theta) = (e^{ij\theta} + e^{-ij\theta})/2 = (\alpha^j + \alpha^{-j})/2$. Then

$$A = \begin{pmatrix} \cos 2\theta & \cos 3\theta & \cos 4\theta & \cdots & \cos(n+1)\theta \\ \cos 3\theta & \cos 4\theta & \cos 5\theta & \cdots & \\ \cos 4\theta & \cos 5\theta & \cdots & & . \\ \cdots & & & & \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha^2+\alpha^{-2} & \alpha^3+\alpha^{-3} & \alpha^4+\alpha^{-4} & \cdots & \alpha^{n+1}+\alpha^{-(n+1)} \\ \alpha^3+\alpha^{-3} & \alpha^4+\alpha^{-4} & \alpha^5+\alpha^{-5} & \cdots & \\ \alpha^4+\alpha^{-4} & \alpha^5+\alpha^{-5} & \cdots & & \\ \cdots & & & & \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \alpha & \alpha^{-1} \\ \alpha^2 & \alpha^{-2} \\ \alpha^3 & \alpha^{-3} \\ \vdots & \vdots \\ \alpha^n & \alpha^{-n} \end{pmatrix} \begin{pmatrix} \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^n \\ \alpha^{-1} & \alpha^{-2} & \alpha^{-3} & \cdots & \alpha^{-n} \end{pmatrix}.$$

Let B denote the $n \times 2$ matrix above. We must evaluate $\det\left(I_n + \frac{1}{2}BB^T\right)$, where I_n is the $n \times n$ identity matrix. Consider the $(n+2) \times (n+2)$ matrix

$$C = \begin{pmatrix} I_n & -\frac{1}{2}B \\ B^T & I_2 \end{pmatrix},$$

specified in block form. We may block-row-reduce C by multiplying the second row on the left by $B/2$, and adding this to the first row. This yields

$$\begin{pmatrix} I_n + \frac{1}{2}BB^T & 0 \\ B^T & I_2 \end{pmatrix}.$$

Then the determinant is obtained from the diagonal blocks:

$$\det C = \det(I_n + \tfrac{1}{2}BB^T)\det(I_2) = \det(I_n + \tfrac{1}{2}BB^T).$$

However, if we block-row-reduce by multiplying the first row of C by $-B^T$ and adding to the second row, we establish that

$$\det(I_n + \tfrac{1}{2}BB^T) = \det(C) = \det(I_2 + \tfrac{1}{2}B^TB).$$

The last matrix is a 2×2 matrix:

$$BB^T = \begin{pmatrix} \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^n \\ \alpha^{-1} & \alpha^{-2} & \alpha^{-3} & \cdots & \alpha^{-n} \end{pmatrix} \begin{pmatrix} \alpha & \alpha^{-1} \\ \alpha^2 & \alpha^{-2} \\ \alpha^3 & \alpha^{-3} \\ \vdots & \vdots \\ \alpha^n & \alpha^{-n} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2 + \alpha^4 + \cdots \alpha^{2n} & n \\ n & \alpha^{-2} + \alpha^{-4} + \cdots + \alpha^{-2n} \end{pmatrix} = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$$

since $\alpha^2 = e^{4\pi i/n}$ is an n-th root of unity not equal to 1 (since $n \geq 3$). Thus

$$\det(I_n + \tfrac{1}{2}BB^T) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}\right) = 1 - \frac{n^2}{4}.$$

B-6 Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$. Show that there exist $s, t \in S$ such that $\gcd(s,t)$ is prime. [Here, $\gcd(a,b)$ denotes the greatest common divisor of a and b.]

Solution. Assume to the contrary that $\gcd(s,t)$ is not prime for any $s, t \in S$. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_m\}$ be the set of primes, in increasing order, that occur as divisors of elements of S. We describe a decomposition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ by successively assigning $p_1, p_2, \ldots,$ to either $\mathcal{P}_1$ or $\mathcal{P}_2$.

Having assigned the primes less than p, where $p \in \mathcal{P}$, consider the subset S_p of S consisting of those $s \in S$ such that p is the largest prime factor of s and all other prime factors of s have been assigned entirely to $\mathcal{P}_1$ or entirely to $\mathcal{P}_2$. If all $s \in S_p$ are divisible by p^2, then assign s to $\mathcal{P}_1$. (In particular, if $S_p = \emptyset$, then p is assigned to $\mathcal{P}_1$.) Otherwise, there cannot exist both s_1 and s_2 in S_p that are not divisible by p^2 and such that, for $i = 1, 2$, the primes less than p that divide s_i are in $\mathcal{P}_i$, for that would imply that $\gcd(s_1, s_2) = p$, a prime. So, in this case, assign p to $\mathcal{P}_1$ if and only if all other prime divisors of some $s \in S_p$ have been assigned to $\mathcal{P}_2$; otherwise, assign p to $\mathcal{P}_2$. Note that, in any case, $p_1 \in \mathcal{P}_1$, so that $\mathcal{P}_1 \neq \emptyset$.

Now set $n = \prod_{p \in \mathcal{P}_1} p$. There exists $s \in S$ such that $\gcd(s,n) = 1$ or $\gcd(s,n) = s$.

Suppose first that $\gcd(s,n) = 1$, so that all prime divisors of s are in $\mathcal{P}_2$. Let p be the largest of these. Since $p \in \mathcal{P}_2$, there exists $t \in S_p$ such that all the prime divisors of t except p are in $\mathcal{P}_1$ and t is not divisible by p^2. But that implies that $\gcd(s,t) = p$, a prime, a contradiction.

Hence s divides n, so all prime divisors of s are in $\mathcal{P}_1$. Let p be the largest of these. Since s divides n, s is not divisible by p^2. However, since $s \in S_p$, this would have caused the assignment of p to $\mathcal{P}_2$, a contradiction.

Therefore, the assumption that $\gcd(s,t)$ is not prime for any $s, t \in S$ is false.

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